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Modeling of acoustic, elastic, and electro-magnetic waves

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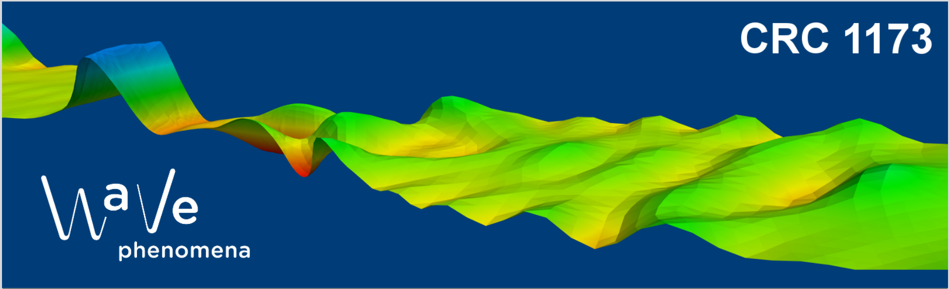

Newton's law

Force F mass m acceleration a

Modeling in continuum mechanics

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Configuration

Select domains in space $\Omega \subset \mathbb{R}^d$ and in time $I \subset \mathbb{R}$,
specify boundary parts $\Gamma_j \subset \partial\Omega$, $j = 1, \dots, m$.

Constituents

Which physical quantities determine the model?
Which quantities directly depend on these quantities?

Parameters

Which material data are required for the model?

Balance relations

Relations between the physical quantities (and external sources)
derived from basic energetic or kinematic principles.

Material laws

Relations between the physical quantities
which have to be determined by measurements.

Boundary and initial data

Additional data on the boundary $\partial(I \times \Omega)$ are required to determine a solution.

The wave equation $\partial_t^2 u - c^2 \partial_x^2 u = 0$ in 1d

Configuration

interval $\Omega = (0, L) \subset \mathbb{R}$ in space, time interval $I = (0, T) \subset \mathbb{R}$.

Constituents

vertical displacement	$u \colon [0, T] \times \overline{\Omega} \longrightarrow \mathbb{R}$	tension	$\sigma \colon [0, T] \times \overline{\Omega} \longrightarrow \mathbb{R}$
velocity	$v = \partial_t u$	strain	$\varepsilon = \partial_x u$
acceleration	$a = \partial_t v = \partial_t^2 u$	strain rate	$\partial_t \varepsilon = \partial_x v$

The displacement describes the position $\big(x, u(t, x)\big) \in \mathbb{R}^2$ at time t.
The tension describes the forces between the points $x \in \Omega$.

Material parameters

mass density ρ, stiffness κ, wave speed $c = \sqrt{\kappa/\rho}$

Newton's law: Balance of momentum ρv

balance relation for all $0 < x_1 < x_2 < L$ and $0 < t_1 < t_2 < T$:

$$\int_{x_1}^{x_2} \rho(x)\big(v(t_2, x) - v(t_1, x)\big)\,\mathrm{d}x = \int_{t_1}^{t_2} \big(\sigma(t, x_2) - \sigma(t, x_1)\big)\,\mathrm{d}t \quad \Longleftrightarrow \quad \rho \partial_t v = \partial_x \sigma$$

Material law

$\sigma = \kappa \varepsilon$

Boundary and initial data

$u(0, x) = u_0(x)$ and $v(0, x) = v_0(x)$ for $x \in \Omega$, $u(t, 0) = u(t, L) = 0$ for $t \in (0, T)$

Harmonic waves $u(t,x) = A\exp\big(\mathrm{i}(kx - \omega t)\big)$

Characteristic quantities

amplitude	A
wave number	k
angular frequency	ω
frequency	$\nu = \omega/2\pi$
wave speed	$c = \omega/k$
wave length	$\lambda = c/\nu$

Interaction with material: anharmonic waves

attenuation	$\omega \to \omega - \mathrm{i}\tau^{-1}$, i.e., $u(t,x) = A\exp(-\tau^{-1}t)\exp\big(\mathrm{i}(kx - \omega t)\big)$
dispersion	$\omega = \omega(k)$

The Maxwell model for viscous waves

Combining a harmonic wave with several anharmonic waves described by the stiffness $\kappa = \kappa_0 + \kappa_1 + \cdots + \kappa_r$ and relaxation times τ_j

$$\sigma_0 = \kappa_0 \varepsilon\,, \qquad \partial_t \sigma_j + \tau_j^{-1}\sigma_j = \kappa_j \partial_t \varepsilon\,, \qquad j = 1,\dots,r$$

results for $\sigma = \sigma_0 + \sigma_1 + \cdots + \sigma_r$ in

$$\rho\partial_t v = \partial_x \sigma\,,\ \partial_t\sigma(t) = \kappa\partial_x v(t) + \int_0^t \dot\kappa(t-s)\partial_x v(s)\,\mathrm{d}s \ \text{ with }\ \dot\kappa(s) = -\sum_{j=1}^r \frac{\kappa_j}{\tau_j}\exp\Big(-\frac{s}{\tau_j}\Big)\,.$$

Elastic waves $\rho\partial_t^2\mathbf{u} - \operatorname{div}\mathbf{C}\varepsilon(\mathbf{u}) = \mathbf{0}$

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Configuration

spatial domain $\Omega \subset \mathbb{R}^3$, time interval $I = (0, T)$, boundary decomposition $\partial\Omega = \Gamma_D \cup \Gamma_S$

Constituents

displacement	$\mathbf{u}\colon [0,T] \times \overline{\Omega} \longrightarrow \mathbb{R}^3$	stress	$\boldsymbol{\sigma}\colon [0,T] \times \overline{\Omega} \longrightarrow \mathbb{R}^{3\times 3}_{\mathrm{sym}}$
velocity	$\mathbf{v} = \partial_t \mathbf{u}$	strain	$\varepsilon(\mathbf{u}) = \operatorname{sym}(\mathrm{D}\mathbf{u}) = \varepsilon$
acceleration	$\mathbf{a} = \partial_t \mathbf{v} = \partial_t^2 \mathbf{u}$	strain rate	$\varepsilon(\mathbf{v}) = \operatorname{sym}(\mathrm{D}\mathbf{v}) = \partial_t \varepsilon$

The displacement describes the position $x + \mathbf{u}(t,x) \in \mathbb{R}^3$ at time t,
the stress describes the force $\boldsymbol{\sigma}\mathbf{n}$ between the material points in direction $\mathbf{n}$.

Material parameters

mass density $\rho\colon \Omega \longrightarrow (0,\infty)$, Hooke's tensor $\mathbf{C}$

Newton's law: balance of momentum $\rho\mathbf{v}$

Balance relation for all $K \subset \Omega$ and $0 < t_1 < t_2 < T$ (without external loads):

$$\int_K \rho(x)\big(\mathbf{v}(t_2,x) - \mathbf{v}(t_1,x)\big)\,\mathrm{d}x = \int_{t_1}^{t_2}\int_{\partial K} \boldsymbol{\sigma}(t,x)\mathbf{n}(x)\,\mathrm{d}a\,\mathrm{d}t \quad\Longleftrightarrow\quad \rho\partial_t\mathbf{v} = \operatorname{div}\boldsymbol{\sigma}$$

Hooke's law: Material law

$\boldsymbol{\sigma} = \mathbf{C}\varepsilon$ (in case of small strains)

Boundary and initial data

$\mathbf{u}(0) = \mathbf{u}_0$, $\mathbf{v}(0) = \mathbf{v}_0$ in Ω, $\mathbf{u}(t) = \mathbf{u}_D(t)$ on Γ_D, $\boldsymbol{\sigma}(t)\mathbf{n} = \mathbf{g}_S$ on Γ_S, $t \in (0,T)$.

Visco-elastic waves

The balance of momentum $\rho\partial_t \mathbf{v} = \operatorname{div} \boldsymbol{\sigma} + \mathbf{f}$ (Newton's law) together with Hooke's law $\boldsymbol{\sigma} = \mathbf{C}\varepsilon(\mathbf{u})$ describes elastic waves. We observe

$$\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}(0) + \int_0^t \partial_t \boldsymbol{\sigma}(s)\,\mathrm{d}s = \boldsymbol{\sigma}(0) + \int_0^t \mathbf{C}\varepsilon(\mathbf{v}(s))\,\mathrm{d}s\,.$$

Linear visco-elastic waves are described by a *retarded material law*

$$\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}(0) + \int_0^t \mathbf{C}(t-s)\varepsilon(\mathbf{v}(s))\,\mathrm{d}s \implies \partial_t\boldsymbol{\sigma}(t) = \mathbf{C}(0)\varepsilon(\mathbf{v}(t)) + \int_0^t \dot{\mathbf{C}}(t-s)\varepsilon(\mathbf{v}(s))\,\mathrm{d}s\,.$$

For *Generalized Standard Linear Solids* the *relaxation tensor* is chosen as

$$\dot{\mathbf{C}}(s) = -\sum_{j=1}^r \frac{1}{\tau_j}\exp\Big(-\frac{s}{\tau_j}\Big)\mathbf{C}_j\,, \qquad \mathbf{C} = \mathbf{C}_0 + \mathbf{C}_1 + \cdots + \mathbf{C}_r\,.$$

Introducing the corresponding stress decomposition $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + \cdots + \boldsymbol{\sigma}_r$ with

$$\boldsymbol{\sigma}_j(t) = \int_0^t \exp\left(\frac{s-t}{\tau_j}\right)\mathbf{C}_j\varepsilon(\mathbf{v}(s))\,\mathrm{d}s\,, \qquad j = 1,\dots,r$$

results in

$$\begin{aligned} \rho\,\partial_t\mathbf{v} - \nabla\cdot(\boldsymbol{\sigma}_0 + \cdots + \boldsymbol{\sigma}_r) &= \mathbf{f}\,, \\ \partial_t\boldsymbol{\sigma}_0 - \mathbf{C}_0\varepsilon(\mathbf{v}) &= \mathbf{0}\,, \\ \partial_t\boldsymbol{\sigma}_j - \mathbf{C}_j\varepsilon(\mathbf{v}) + \tau_j^{-1}\boldsymbol{\sigma}_j &= \mathbf{0}\,, \qquad j = 1,\dots,r\,. \end{aligned}$$

Acoustic waves in solids $\partial_t^2 p - c^2 \Delta p = 0$

In isotropic media, Hooke's tensor

$$\mathbf{C}\varepsilon = 2\mu\varepsilon + \lambda\,\mathrm{trace}(\varepsilon)\mathbf{I} = 2\mu\,\mathrm{dev}(\varepsilon) + \kappa\,\mathrm{trace}(\varepsilon)\mathbf{I}\,, \qquad \mathrm{dev}(\varepsilon) = \varepsilon - \frac{1}{3}\mathrm{trace}(\varepsilon)\mathbf{I}$$

depends on the shear modulus μ and the compression modulus $\kappa = \frac{2}{3}\mu + \lambda$, i.e.,

$$\partial_t^2 \mathbf{u} + \mu \nabla \times \nabla \times \mathbf{u} - 3\kappa \nabla(\nabla \cdot \mathbf{u}) = \mathbf{f}\,.$$

Vanishing shear modulus $\mu \to 0$ gives for the *hydrostatic pressure* $p = \frac{1}{3}\mathrm{trace}(\boldsymbol{\sigma})$

$$\rho\,\partial_t \mathbf{v} - \nabla p = \mathbf{f}\,, \qquad \partial_t p - \kappa \nabla \cdot \mathbf{v} = 0\,.$$

In homogeneous media, this yields (in case of $\mathbf{f} = \mathbf{0}$)

$$\partial_t^2 p - c^2 \Delta p = 0\,, \qquad c = \sqrt{\kappa/\rho}\,.$$

Visco-acoustic waves

$$\partial_t p(t) = \kappa \nabla \cdot \mathbf{v}(t) + \int_0^t \dot\kappa(t-s) \nabla \cdot \mathbf{v}(s)\,\mathrm{d}s\,, \qquad \dot\kappa(s) = -\sum_{j=1}^r \frac{\kappa_j}{\tau_j} \exp\Big(-\frac{s}{\tau_j}\Big)$$

with $\kappa = \kappa_0 + \kappa_1 + \cdots + \kappa_r$ yields

$$\begin{aligned} \rho\,\partial_t \mathbf{v} - \nabla(p_0 + \cdots + p_r) &= \mathbf{f}\,, \\ \partial_t p_0 - \kappa_0 \nabla \cdot \mathbf{v} &= 0\,, \\ \partial_t p_j - \kappa_j \nabla \cdot \mathbf{v} + \tau_j^{-1} p_j &= 0\,, \qquad j = 1, \dots, r\,. \end{aligned}$$

Electro-magnetic waves $\partial_t^2 \mathbf{E} - c^2 \nabla \times \nabla \times \mathbf{E} = \mathbf{0}$

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Configuration

spatial domain $\Omega \subset \mathbb{R}^3$, time interval $I = (0, T)$, boundary $\partial\Omega = \Gamma_{\mathrm{E}} \cup \Gamma_{\mathrm{I}}$

Constituents

electric field	$\mathbf{E}\colon \overline{I \times \Omega} \to \mathbb{R}^3$	magnetic field intensity	$\mathbf{H}\colon \overline{I \times \Omega} \to \mathbb{R}^3$
electric flux density	$\mathbf{D}\colon \overline{I \times \Omega} \to \mathbb{R}^3$	magnetic induction	$\mathbf{B}\colon \overline{I \times \Omega} \to \mathbb{R}^3$
electric current density	$\mathbf{J}\colon I \times \Omega \to \mathbb{R}^3$	electric charge density	$\rho\colon I \times \Omega \to \mathbb{R}$

Balance relations by Faraday, Ampere, and Gauß

For all $0 < t_1 < t_2 < T$ and (sufficiently smooth) volumes and surfaces $K, A \subset \Omega$:

$$\int_A \big(\mathbf{B}(t_2) - \mathbf{B}(t_1)\big) \cdot \mathrm{d}a = -\int_{t_1}^{t_2} \int_{\partial A} \mathbf{E} \cdot \mathrm{d}\ell \, \mathrm{d}t \qquad \Longrightarrow \quad \partial_t \mathbf{B} + \nabla \times \mathbf{E} = \mathbf{0}$$

$$\int_A \big(\mathbf{D}(t_2) - \mathbf{D}(t_1)\big) \cdot \mathrm{d}a = \int_{t_1}^{t_2} \left(\int_{\partial A} \mathbf{H} \cdot \mathrm{d}\ell - \int_A \mathbf{J} \cdot \mathrm{d}a \right) \mathrm{d}t \qquad \Longrightarrow \quad \partial_t \mathbf{D} - \nabla \times \mathbf{H} = -\mathbf{J}$$

$$\int_{\partial K} \mathbf{B} \cdot \mathrm{d}a = 0 \qquad \Longrightarrow \quad \nabla \cdot \mathbf{B} = 0$$

$$\int_{\partial K} \mathbf{D} \cdot \mathrm{d}a = \int_K \rho \, \mathrm{d}x \qquad \Longrightarrow \quad \nabla \cdot \mathbf{D} = \rho$$

Material laws in vacuum

$\mathbf{D} = \varepsilon_0 \mathbf{E}$, $\mathbf{B} = \mu_0 \mathbf{H}$, $\mathbf{J} = \mathbf{0}$, $\rho = 0$, $c = 1/\sqrt{\varepsilon_0 \mu_0}$

Electro-magnetic waves in matter

Material data

permittivity ε_0, permeability μ_0, susceptibility χ, conductivities σ, ζ

Material laws

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}(\mathbf{E}, \mathbf{B}) \qquad \mathbf{P} \text{ polarization}$$

$$\mu_0 \mathbf{H} = \mathbf{B} - \mathbf{M}(\mathbf{E}, \mathbf{B}) \qquad \mathbf{M} \text{ magnetization}$$

electric current density $\mathbf{J} = \sigma(\mathbf{E}, \mathbf{H})\mathbf{E} + \mathbf{J}_0$

linear materials with instantaneous response: $\mathbf{P} = \varepsilon_0 \chi \mathbf{E} \;\Rightarrow\; \mathbf{D} = \varepsilon_\mathrm{r} \mathbf{E}$, $\varepsilon_\mathrm{r} = \varepsilon_0(1 + \chi)$

linear materials with retarded response: $\mathbf{P}(t) = \varepsilon_0 \int_{-\infty}^{t} \chi(t - s)\mathbf{E}(s)\,\mathrm{d}s$

nonlinear materials

$$\mathbf{P}(t) = \varepsilon_0 \int_{-\infty}^{t} \chi_1(t-s)\mathbf{E}(s)\,\mathrm{d}s + \int_{-\infty}^{t}\int_{-\infty}^{t}\int_{-\infty}^{t} \chi_3(t-s_1, t-s_2, t-s_3)\big(\mathbf{E}(s_1), \mathbf{E}(s_2), \mathbf{E}(s_3)\big)\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}s_3$$

materials of Kerr-type: $\mathbf{P} = \chi_1 \mathbf{E} + \chi_3 |\mathbf{E}|^2 \mathbf{E}$

Boundary conditions

perfectly conducting boundary $\quad \mathbf{E} \times \mathbf{n} = \mathbf{0}$ and $\mathbf{B} \cdot \mathbf{n} = 0$ on Γ_E

impedance (or Silver–Müller) boundary $\quad \mathbf{H} \times \mathbf{n} + \big(\zeta(\mathbf{E} \times \mathbf{n})\mathbf{E} \times \mathbf{n}\big) \times \mathbf{n} = \mathbf{0}$ on Γ_I