**Lecture Notes for the MFO seminar on wave phenomena**

# Space-time approximations for linear acoustic, elastic, and electro-magnetic wave equations

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## 1. Modeling of acoustic, elastic, and electro-magnetic waves

Mathematical modeling of physical processes yields a system of partial differential equations that describes the behavior of a system physically correct and allows for analytical and numerical predictions of the system behavior. Here we start by shortly summarizing modeling principles which are illustrated for simple linear models in one space dimension. Then this is specified for different types of wave equations.

### 1.1. Modeling in continuum mechanics

Describing a model in continuum mechanics is a complex process combining physical principles, parameters and data. For a mathematical framework, we introduce the following terminology:

- **Geometric configuration**
  We select a domain in space $\Omega \subset \mathbb{R}^d$ ($d \in \{1, 2, 3\}$) and a time interval $I \subset \mathbb{R}$, and for the specification of boundary conditions we select boundary parts $\Gamma_k \subset \partial\Omega$, $k = 1, \dots, m$, where $m$ is the number of components of the variables which describe the current state of the physical system.
- **Constituents**
  Which physical quantities determine the model?
  Which quantities directly depend on these primary quantities?
  For the mathematical formulation it is required to select a set of primary variables.
- **Parameters**
  Which material data are required for the model?
  Which properties do these parameters have in order to be physically meaningful?
- **Balance relations**
  This collects relations between the physical quantities (and external sources) which are derived from basic energetic or kinematic principles. These relations are independent of specific materials and applications.
- **Material laws**
  This collects relations between the physical quantities that have to be determined by measurements and depend on the specific material and application.
- **External forces, boundary and initial data**
  The system behavior is controlled by the initial state at $t = 0$, by external forces in the interior of the space-time domain $I \times \Omega$, and by conditions on the boundary $I \times \partial\Omega$.

### 1.2. The wave equation in 1d

This formalism is now specified for the most simple wave model in 1d with constant coefficients. Therefore, we assume that all quantities are sufficiently smooth, so that all derivatives and integrals are well-defined.

**Configuration.** We consider an interval $\Omega = (0, X) \subset \mathbb{R}$ in space and a time interval $I = (0, T) \subset \mathbb{R}$ for given $X, T > 0$.

**Constituents.** Here, we consider the simplified situation that material points in $\mathbb{R}^2$ move up and down vertically. The state of this physical system is then determined by the vertical *displacement*

$$u\colon [0,T] \times \overline{\Omega} \longrightarrow \mathbb{R}$$

describing the position of the material point $\big(x, u(t,x)\big) \in \mathbb{R}^2$ at time $t$, and the *tension*

$$\sigma\colon [0,T] \times \overline{\Omega} \longrightarrow \mathbb{R}$$

describing the forces between the points $x \in \Omega$. In this simplified 1d setting with vertical displacements the tension corresponds to the *shear stress* in higher dimensions.
Depending on the primal variable $u$, we define the *velocity* $v = \partial_t u$, the *acceleration* $a = \partial_t v = \partial_t^2 u$, the *strain* $\varepsilon = \partial_x u$, and the *strain rate* $\partial_t \varepsilon = \partial_x v = \partial_x \partial_t u$.

**Material parameters.** This simple model only depends on the *mass density* $\rho > 0$ and the *stiffness* $\kappa > 0$; together, this defines $c = \sqrt{\kappa/\rho}$. We will see that $c$ is the wave speed which characterizes this model.

**Balance of momentum.** Depending on the velocity $v$ and the mass density $\rho$ we define the *momentum* $\rho v$. Newton's law states that the temporal change of the momentum in time equals the sum of all driving forces. Here, without any external forces, this balance relation reads as follows:
for all $0 < x_1 < x_2 < X$ and $0 < t_1 < t_2 < T$ we have

$$\int_{x_1}^{x_2} \rho(x)\big(v(t_2,x) - v(t_1,x)\big)\,\mathrm{d}x = \int_{t_1}^{t_2} \big(\sigma(t,x_2) - \sigma(t,x_1)\big)\,\mathrm{d}t\,.$$

For smooth functions this yields

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2} \rho(x)\partial_t v(t,x)\,\mathrm{d}t\,\mathrm{d}x = \int_{t_1}^{t_2}\int_{x_1}^{x_2} \partial_x\sigma(t,x)\,\mathrm{d}x\,\mathrm{d}t\,,$$

and since this holds for all $0 < x_1 < x_2 < X$ and $0 < t_1 < t_2 < T$, this holds point-wise, i.e.,

$$\rho(x)\partial_t v(t,x) = \partial_x\sigma(t,x)\,, \qquad (t,x) \in (0,T)\times(0,X)\,. \tag{1}$$

**Material law.** One observes that the tension $\sigma(t,x)$ only depends on the strain $\varepsilon(t,x) = \partial_x u(t,x)$. This is formulated as a material law: a material is by definition *elastic*, if a function $\Sigma$ exists such that $\sigma = \Sigma(\partial_x u)$, and it is *linear elastic*, if $\sigma = \kappa\varepsilon$ with stiffness $\kappa > 0$. In a homogeneous material, the stiffness $\kappa$ is independent of $x \in (0, X)$.

**Boundary and initial data.** The actual physical state at time $t$ of the system depends on its state at the beginning $t = 0$ and on constraints at the boundary. Here, assume that at $t = 0$ the system is given by the initial displacement $u(0,x) = u_0(x)$ and velocity $v(0,x) = v_0(x)$ for $x \in \Omega$, and we use homogeneous boundary conditions $u(t,0) = u(t,X) = 0$ for $t \in [0,T]$ corresponding to a string that is fixed at the endpoints.
Inserting $v = \partial_t u$ and $\varepsilon = \partial_x u$ in (1) we obtain the second-order formulation of the wave equation

$$\begin{aligned}
\partial_t^2 u(t,x) - c^2\partial_x^2 u(t,x) &= 0 && \text{for } (t,x) \in (0,T)\times(0,X)\,, && \text{(2a)}\\
u(0,x) &= u_0(x) && \text{for } x \in (0,X) \text{ at } t = 0\,, && \text{(2b)}\\
\partial_t u(0,x) &= v_0(x) && \text{for } x \in (0,X) \text{ at } t = 0\,, && \text{(2c)}\\
u(t,x) &= 0 && \text{for } x \in \{0,X\} \text{ and } t \in (0,T)\,. && \text{(2d)}
\end{aligned}$$

Note that the same equation can be derived for a 1d wave with horizontal displacement, corresponding to an actual position of the material point $x + u(x) \in \mathbb{R}$.

**The solution of the linear wave equation in 1d in homogeneous media.** The equation (2) with constant wave speed $c > 0$ can be solved explicitly. For given initial values (2b) and (2c) the solution is given within the cone

$$\mathcal{C} = \Big\{(t,x) \in (0,T) \times (0,X) \colon 0 < x - ct < x + ct < X\Big\}$$

by the d'Alembert formula

$$u(t,x) = \frac{1}{2}\left(u_0(x-ct) + u_0(x+ct) + \frac{1}{c}\int_{x-ct}^{x+ct} v_0(\xi)\,\mathrm{d}\xi\right), \quad (t,x) \in \mathcal{C}\,.$$

Now we consider the solution in the bounded interval $\Omega = (0,X)$ of length $X = \pi$ with homogeneous Dirichlet boundary conditions (2d). The solution can be expanded into eigenmodes of the operator $-\partial_x^2 u$ in $\mathrm{H}^1_0(\Omega)\cap\mathrm{H}^2(\Omega)$, so that we obtain

$$u(t,x) = \sum_{n=1}^{\infty}\Big(\alpha_n \cos(cnt) + \beta_n \sin(cnt)\Big)\sin(nx)\,,$$

where the coefficients are determined by the initial values (2b) and (2c). For the special example with initial values $u_0(x) = 1$, $v_0(x) = 0$ for $x \in (0,\pi)$, and wave speed $c = 1$, we obtain the explicit Fourier representation

$$u(t,x) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\cos\big((2n+1)t\big)\sin\big((2n+1)x\big) = \frac{1}{2}\Big(u_0(x+t) + u_0(x-t)\Big)\,, \tag{3}$$

where the initial function $u_0$ is extended to the periodic function

$$u_0(x) = \begin{cases} \phantom{-}1 & x \in (0,\pi) + 2\pi\mathbb{Z}\,,\\ \phantom{-}0 & x \in \pi\mathbb{Z}\,,\\ -1 & x \in (-\pi,0) + 2\pi\mathbb{Z}\,,\end{cases}$$

cf. Fig. 1. We observe that this solution solves the wave equation only in a weak sense since it is discontinuous along linear characteristics $x \pm ct = \mathrm{const}$.

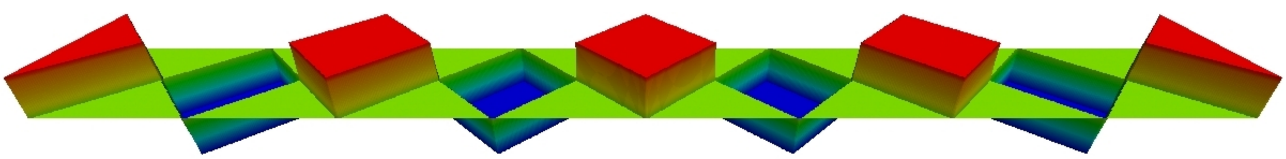


FIGURE 1. Weak solution $u \in \mathrm{L}_2\big((0,8)\times(0,\pi)\big)$ with initial values for $u(0,\cdot) = 1$, $\partial_t u(0,\cdot) = 0$, and homogeneous Dirichlet boundary values $u(\cdot,0) = u(\cdot,\pi) = 0$.

### 1.3. **Harmonic, anharmonic and viscous waves**

Special solutions of the linear wave equation (2) can be derived by the ansatz

$$u(t,x) = \exp(-\mathrm{i}\omega t)a(x)$$

with a fixed frequency $\omega \in \mathbb{R}$. This yields in case of constant wave speed $c = \sqrt{\kappa/\rho}$

$$\partial_t^2 u(t,x) - c^2\partial_x^2 u(t,x) = -\Big(\omega^2 a(x) + c^2\partial_x^2 a(x)\Big)\exp(-\mathrm{i}\omega t)\,.$$

The equation $\omega^2 a(x) + c^2\partial_x^2 a(x) = 0$ is solved by $a(x) = a_0\exp(\mathrm{i}kx)$ with $k = \omega/c$ and $a_0 \in \mathbb{R}$, cf. Tab. 1.

TABLE 1. Characteristic quantities for harmonic waves $u(t,x) = a_0\exp\big(\mathrm{i}(kx-\omega t)\big)$.

| | | | | | |
|---|---|---|---|---|---|
| wave number | $k$ | angular frequency | $\omega$ | frequency | $\nu = \omega/2\pi$ |
| wave speed | $c = \omega/k$ | wave length | $\lambda = c/\nu$ | amplitude | $a_0$ |

**Interaction with material: anharmonic waves.** The harmonic wave with constant amplitude is an idealistic model. This contradicts to observations: a wave traveling through material interacts with the particles in some sense, so that the amplitude is decreasing in time. A simple ansatz are waves of the form

$$u(t,x) = a(t)\exp\big(\mathrm{i}(kx-\omega t)\big)\,,\qquad a(t) = a_0\exp(-\tau t) \tag{4}$$

depending on wave number $k$, angular frequency $\omega$, and relaxation time $\tau > 0$. Then, we observe for (4) in case of constant $\rho$ and $\kappa$

$$(\rho\partial_t^2 - \kappa\partial_x^2)u(x,t) = \big(\rho(\tau+\mathrm{i}\omega)^2 + \kappa k^2\big)u(x,t)\,,\qquad \partial_t u(x,t) = -(\tau+\mathrm{i}\omega)u(x,t)$$

which yields with the angular frequency

$$\omega = \sqrt{k^2\kappa/\rho + \tau^2} \in \mathbb{R} \tag{5}$$

a solution of the wave equation with attenuation

$$\rho\partial_t^2 u(t,x) - \kappa\partial_x^2 u(t,x) + 2\tau\rho\,\partial_t u(t,x) = 0\,. \tag{6}$$

In general, one observes that the wave speed depends on the frequency of the wave, i.e., the wave is *dispersive*. For the case of constant parameters this is characterized by the *dispersion relation* $\omega = \omega(k)$. In this example, we find the dispersion relation (5) for the wave equation with attenuation (6). For the general description of real media this approach is too simple and applies only for the wave propagation within a limited frequency range, in particular since the relaxation time also depends on the frequency. For viscous waves suitable material laws are constructed where the parameters can be determined from measurements of the dispersion relation at sample frequencies which are relevant for the application. This is now demonstrated for a specific example.

**A model for viscous waves.** One approach to characterize waves with dispersion is to use a linear superposition of the constitutive law for a harmonic wave with several relations for anharmonic waves. In this ansatz the material law for the stress is based on a decomposition $\sigma = \sigma_0 + \sigma_1 + \cdots + \sigma_r$ with Hooke's law for $\sigma_0$, i.e.,

$$\sigma_0 = \kappa_0\varepsilon\,, \tag{7a}$$

and several Maxwell bodies for $\sigma_1,\dots,\sigma_r$ described by the relations

$$\partial_t\sigma_j + \tau_j^{-1}\sigma_j = \kappa_j\partial_t\varepsilon\,,\qquad j = 1,\dots,r\,. \tag{7b}$$

This model depends on the stiffness of the components $\kappa_0,\dots,\kappa_r$ and relaxation times $\tau_1,\dots,\tau_r$. Solving the linear ODE (7b) with initial value $\sigma_j(0) = 0$ and inserting $\partial_t\varepsilon = \partial_x v$ yields

$$\sigma_j(t) = \int_0^t \kappa_j\exp\Big(-\frac{1}{\tau_j}(t-s)\Big)\partial_x v(s)\,\mathrm{d}s\,,$$

and together with (7a) we obtain the *retarded material law*

$$\sigma(t) = \kappa_0 \partial_x u(t) + \int_0^t \sum_{j=1}^r \kappa_j \exp\Big(-\frac{1}{\tau_j}(t-s)\Big)\partial_x v(s)\,\mathrm{d}s\,.$$

This can be summarized to

$$\begin{aligned}
\partial_t \sigma(t) &= \kappa_0 \partial_x v(t) + \sum_{j=1}^r \kappa_j \partial_x v(t) - \int_0^t \sum_{j=1}^r \frac{\kappa_j}{\tau_j} \exp\Big(-\frac{1}{\tau_j}(t-s)\Big)\partial_x v(s)\,\mathrm{d}s\\
&= \kappa \partial_x v(t) + \int_0^t \dot\kappa(t-s)\partial_x v(s)\,\mathrm{d}s
\end{aligned}$$

with the total stiffness $\kappa = \kappa_0 + \kappa_1 + \cdots + \kappa_r$ and the *retardation kernel*

$$\dot\kappa(s) = -\sum_{j=1}^r \frac{\kappa_j}{\tau_j}\exp\Big(-\frac{s}{\tau_j}\Big)\,.$$

Together with the balance relation $\rho \partial_t v = \partial_x \sigma$ this is a model for viscous waves.

### 1.4. **Elastic waves**

In the next step we derive equations for waves in solids. We consider heterogeneous media where the material parameters depend on the position, and we assume that the wave energy is sufficiently small, so that the material law can be approximated by a linear relation.

**Configuration.** We consider an elastic body in the spatial domain $\Omega \subset \mathbb{R}^3$ and we fix a time interval $I = (0,T)$. The boundary $\partial\Omega = \Gamma_{\mathrm{V}} \cup \Gamma_{\mathrm{S}}$ is decomposed into parts corresponding to dynamic boundary conditions for the velocity and static boundary conditions for the stress.

**Constituents.** The current state of the body is described by the *deformation* or by the *displacement*

$$\boldsymbol{\varphi} = \mathrm{id} + \mathbf{u}\colon [0,T]\times\overline{\Omega} \longrightarrow \mathbb{R}^3\,, \qquad \mathbf{u}\colon [0,T]\times\overline{\Omega} \longrightarrow \mathbb{R}^3\,,$$

i.e., $\boldsymbol{\varphi}(t,\mathbf{x}) = \mathbf{x} + \mathbf{u}(t,\mathbf{x})$ is the actual position of the point $\mathbf{x}\in\Omega$ at time $t$. Depending on the displacement, we define the *velocity* $\mathbf{v} = \partial_t \mathbf{u}$, the *strain* $\boldsymbol{\varepsilon}(\mathbf{u}) = \mathrm{sym}(\mathrm{D}\mathbf{u})$, the *acceleration* $\mathbf{a} = \partial_t \mathbf{v} = \partial_t^2 \mathbf{u}$, and the *strain rate* $\boldsymbol{\varepsilon}(\mathbf{v}) = \mathrm{sym}(\mathrm{D}\mathbf{v}) = \partial_t \boldsymbol{\varepsilon}(\mathbf{u})$.
The internal forces in the material are described by the *stress tensor*

$$\boldsymbol{\sigma}\colon [0,T]\times\overline{\Omega} \longrightarrow \mathbb{R}^{3\times 3}_{\mathrm{sym}}\,.$$

**Material parameters.** Measurements are required to determine the distribution of the mass density
$$\rho\colon \overline{\Omega} \longrightarrow (0,\infty)$$

and to determine the material stiffness in all directions which are collected in Hooke's tensor
$$\mathbf{C}\colon \overline{\Omega} \longrightarrow \mathcal{L}(\mathbb{R}^{3\times 3}_{\mathrm{sym}}, \mathbb{R}^{3\times 3}_{\mathrm{sym}})\,.$$

**Balance of momentum.** Newton's law postulates equality between the temporal change of the momentum $\rho\mathbf{v}$ in any time interval $(t_1,t_2) \in (0,T)$ within any subvolume $K \subset \Omega$ and the driving forces on the boundary $\partial K$ described by the stress in direction of the outer normal vector $\mathbf{n}$ on $\partial K$. This results in the balance relation (without external loads)

$$\int_K \rho(\mathbf{x})\big(\mathbf{v}(t_2,\mathbf{x}) - \mathbf{v}(t_1,\mathbf{x})\big)\ \mathrm{d}\mathbf{x} = \int_{t_1}^{t_2}\int_{\partial K} \boldsymbol{\sigma}(t,\mathbf{x})\mathbf{n}(\mathbf{x})\ \mathrm{d}\mathbf{a}\ \mathrm{d}t\,. \tag{8}$$

For smooth functions we obtain by the Gauß theorem

$$\int_K \int_{t_1}^{t_2} \rho(\mathbf{x})\partial_t \mathbf{v}(t_2,\mathbf{x})\ \mathrm{d}t\ \mathrm{d}\mathbf{x} = \int_{t_1}^{t_2}\int_K \operatorname{div}\boldsymbol{\sigma}(t,\mathbf{x})\ \mathrm{d}\mathbf{x}\ \mathrm{d}t\,,$$

and since this holds for all time intervals and subvolumes, we get the pointwise relation

$$\rho\partial_t\mathbf{v} = \operatorname{div}\boldsymbol{\sigma} \qquad \text{in } (0,T)\times\Omega\,. \tag{9}$$

**Remark 1.** In the balance relation (8) only the normal stress $\boldsymbol{\sigma}(t,\mathbf{x})\mathbf{n}$ for all directions $\mathbf{n} \in S^2$ on the boundary of a subvolume $K \subset \Omega$ is included. This described the force between material points left and right from $x \in \partial K$ with respect to the direction $\mathbf{n}$. The existence of such a vector for all directions and all points is postulated by the Cauchy axiom, and by the Cauchy theorem a tensor representing this force exists; moreover, the symmetry of this tensor is a consequence of the balance of angular momentum.

**Material law.** Since the forces between the material points $\mathbf{x}_1$ and $\mathbf{x}_2$ only depend on the difference of the actual positions $\mathbf{u}(t,\mathbf{x}_2) - \mathbf{u}(t,\mathbf{x}_1)$, the stress $\boldsymbol{\sigma}(t,\mathbf{x})$ only depends on the deformation gradient $\mathrm{D}\boldsymbol{\varphi}$.
By definition, a material is *elastic*, if a function $\Sigma$ exists such that $\boldsymbol{\sigma} = \Sigma(\mathrm{D}\boldsymbol{\varphi})$. Then, $\partial_t\boldsymbol{\sigma} = \mathrm{D}\Sigma(\mathrm{D}\boldsymbol{\varphi})[\mathrm{D}\mathbf{v}]$. In the limit of small strains the material response can be approximated by a linear model, i.e., we assume $\mathrm{D}\boldsymbol{\varphi} \approx \mathbf{I}$, and we use the linear relation $\partial_t\boldsymbol{\sigma} = \mathrm{D}\Sigma(\mathbf{I})[\mathrm{D}\mathbf{v}]$. In addition, we assume that the stress response is *objective*, i.e., it is independent of the observer's position; then it can be shown that it only depends on the symmetric strain $\boldsymbol{\varepsilon}(\mathbf{u}) = \mathrm{sym}(\mathrm{D}\mathbf{u})$. Together, we obtain Hooke's law

$$\partial_t\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}(\mathbf{v})\,. \tag{10}$$

**Boundary and initial data.** We start with $\mathbf{u}(0) = \mathbf{u}_0$ and $\mathbf{v}(0) = \mathbf{v}_0$ in $\Omega$ at $t = 0$, and for $t \in (0,T)$ we use the boundary conditions for the displacement $\mathbf{u}(t) = \mathbf{u}_{\mathrm{V}}(t)$ or the velocity $\mathbf{v}(t) = \mathbf{v}_{\mathrm{V}}(t)$ on the dynamic boundary $\Gamma_{\mathrm{V}}$, and for the stress $\boldsymbol{\sigma}(t)\mathbf{n} = \mathbf{g}_{\mathrm{S}}$ on the static boundary $\Gamma_{\mathrm{S}}$.

Including external body forces $\mathbf{f}$, we obtain the second-order formulation of the linear wave equation

$$\rho\partial_t^2\mathbf{u} - \operatorname{div}\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u}) = \mathbf{f} \qquad \text{in } (0,T)\times\Omega\,, \tag{11a}$$
$$\mathbf{u}(0) = \mathbf{u}_0 \qquad \text{in } \Omega \text{ at } t = 0\,, \tag{11b}$$
$$\partial_t\mathbf{u}(0) = \mathbf{v}_0 \qquad \text{in } \Omega \text{ at } t = 0\,, \tag{11c}$$
$$\mathbf{u}(t) = \mathbf{u}_{\mathrm{V}}(t) \qquad \text{on } \Gamma_{\mathrm{V}} \text{ for } t \in (0,T)\,, \tag{11d}$$
$$\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u})\mathbf{n} = \mathbf{g}_{\mathrm{S}}(t) \qquad \text{on } \Gamma_{\mathrm{S}} \text{ for } t \in (0,T)\,. \tag{11e}$$

and, equivalently, the first-order formulation

$$\rho\partial_t\mathbf{v} - \operatorname{div}\boldsymbol{\sigma} = \mathbf{f} \qquad \text{in } (0,T)\times\Omega\,, \tag{12a}$$
$$\partial_t\boldsymbol{\sigma} - \mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u}) = \mathbf{0} \qquad \text{in } (0,T)\times\Omega\,, \tag{12b}$$
$$\mathbf{v}(0) = \mathbf{v}_0 \qquad \text{in } \Omega \text{ at } t = 0\,, \tag{12c}$$
$$\boldsymbol{\sigma}(0) = \mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u}_0) \qquad \text{in } \Omega \text{ at } t = 0\,, \tag{12d}$$
$$\mathbf{v}(t) = \partial_t\mathbf{u}_{\mathrm{V}}(t) \qquad \text{on } \Gamma_{\mathrm{V}} \text{ for } t \in (0,T)\,, \tag{12e}$$
$$\boldsymbol{\sigma}(t)\mathbf{n} = \mathbf{g}_{\mathrm{S}}(t) \qquad \text{on } \Gamma_{\mathrm{S}} \text{ for } t \in (0,T)\,. \tag{12f}$$

### 1.5. Visco-elastic waves

The balance of momentum (9) together with Hooke's law $\partial_t\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}(\mathbf{v})$ describes linear elastic waves. We observe

$$\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}(0) + \int_0^t \partial_t\boldsymbol{\sigma}(s)\,\mathrm{d}s = \boldsymbol{\sigma}(0) + \int_0^t \mathbf{C}\boldsymbol{\varepsilon}(\mathbf{v}(s))\,\mathrm{d}s\,.$$

General linear visco-elastic waves are described by a *retarded material law*

$$\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}(0) + \int_0^t \mathbf{C}(t-s)\boldsymbol{\varepsilon}(\mathbf{v}(s))\,\mathrm{d}s$$

implying

$$\partial_t\boldsymbol{\sigma}(t) = \mathbf{C}(0)\boldsymbol{\varepsilon}(\mathbf{v}(t)) + \int_0^t \dot{\mathbf{C}}(t-s)\boldsymbol{\varepsilon}\big(\mathbf{v}(s)\big)\;\mathrm{d}s$$

with a time-dependent extension $\dot{\mathbf{C}}$ of the elasticity tensor $\mathbf{C}$.
In analogy to the 1d model (7), one defines *Generalized Standard Linear Solids* with the *relaxation tensor*

$$\dot{\mathbf{C}}(s) = -\sum_{j=1}^{r} \frac{1}{\tau_j}\exp\Big(-\frac{s}{\tau_j}\Big)\mathbf{C}_j\,, \qquad \mathbf{C}(0) = \mathbf{C}_0 + \mathbf{C}_1 + \dots + \mathbf{C}_r\,.$$

Introducing the corresponding stress decomposition $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + \dots + \boldsymbol{\sigma}_r$ with

$$\boldsymbol{\sigma}_j(t) = \int_0^t \exp\left(\frac{s-t}{\tau_j}\right)\mathbf{C}_j\boldsymbol{\varepsilon}(\mathbf{v}(s))\,\mathrm{d}s\,, \qquad j = 1,\dots,r,$$

results in the first-order system for visco-elastic waves

$$\rho\,\partial_t\mathbf{v} - \nabla\cdot\big(\boldsymbol{\sigma}_0 + \dots + \boldsymbol{\sigma}_r\big) = \mathbf{f}\,, \tag{13a}$$

$$\partial_t\boldsymbol{\sigma}_0 - \mathbf{C}_0\boldsymbol{\varepsilon}(\mathbf{v}) = \mathbf{0}\,, \tag{13b}$$

$$\partial_t\boldsymbol{\sigma}_j - \mathbf{C}_j\boldsymbol{\varepsilon}(\mathbf{v}) + \tau_j^{-1}\boldsymbol{\sigma}_j = \mathbf{0}\,, \qquad j = 1,\dots,r\,. \tag{13c}$$

This is complemented by initial and boundary conditions for the velocity $\mathbf{v}$ and the total stress $\boldsymbol{\sigma}$, which are the observable quantities. The stress components $\boldsymbol{\sigma}_1,\dots,\boldsymbol{\sigma}_r$ are inner variables describing the retarded material law; they can be replaced, e.g., by memory variables encoding the material history.

## 1.6. **Acoustic waves in solids**

In isotropic media, Hooke's tensor only depends on two parameters, e.g., the *Lamé parameters* $\mu, \lambda$

$$\begin{aligned} \mathbf{C}\boldsymbol{\varepsilon} &= 2\mu\boldsymbol{\varepsilon} + \lambda\,\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} \\ &= 2\mu\,\mathrm{dev}(\boldsymbol{\varepsilon}) + \kappa\,\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I}\,, \qquad \mathrm{dev}(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon} - \frac{1}{3}\,\mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I}\,. \end{aligned}$$

For the wave dynamics, one uses a decomposition into components corresponding to shear waves depending on the *shear modulus* $\mu$, and compressional waves depending on the *compression modulus* $\kappa = \frac{2}{3}\mu + \lambda$. Then, the linear second order elastic wave equation (11a) in isotropic and homogeneous media takes the form

$$\rho\partial_t^2\mathbf{u} + \mu\nabla\times\nabla\times\mathbf{u} - 3\kappa\nabla(\nabla\cdot\mathbf{u}) = \mathbf{f}\,.$$

A vanishing shear modulus $\mu \to 0$ leads to the *linear acoustic wave equation* for the *hydrostatic pressure* $p = \frac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma})$ and the velocity, described by the first-order system

$$\begin{aligned} \rho\,\partial_t\mathbf{v} - \nabla p &= \mathbf{f} && \text{in } (0,T)\times\Omega\,, && (14a)\\ \partial_t p - \kappa\nabla\cdot\mathbf{v} &= 0 && \text{in } (0,T)\times\Omega\,, && (14b)\\ \mathbf{v}(0) &= \mathbf{v}_0 && \text{in } \Omega \text{ at } t=0\,, && (14c)\\ p(0) &= p_0 && \text{in } \Omega \text{ at } t=0\,, && (14d)\\ \mathbf{n}\cdot\mathbf{v}(t) &= g_{\mathrm{V}}(t) && \text{on } \Gamma_{\mathrm{V}} \text{ for } t\in(0,T)\,, && (14e)\\ p(t) &= p_{\mathrm{S}}(t) && \text{on } \Gamma_{\mathrm{S}} \text{ for } t\in(0,T)\,, && (14f) \end{aligned}$$

where we set $p_{\mathrm{S}} = \mathbf{n}\cdot\mathbf{g}_{\mathrm{S}}$ for the static boundary condition and $g_{\mathrm{V}} = \mathbf{n}\cdot\mathbf{v}_{\mathrm{V}}$ for the dynamic boundary condition. For acoustics, this corresponds to Dirichlet and Neumann boundary conditions, for elasticity this is reversed. In homogeneous media and for $\mathbf{f} = \mathbf{0}$, (14a) and (14b) combine to the linear second-order acoustic wave equation

$$\partial_t^2 p - c^2\Delta p = 0\,, \qquad c = \sqrt{\kappa/\rho}\,.$$

**Remark 2.** Simply neglecting the shear component is only an approximation and not fully realistic for waves in solids, in particular since by reflections compressional waves split in compressional and shear components. Nevertheless, in applications the acoustic wave equation is used also in solids since the system is much smaller so that computations are much faster.

**Remark 3.** One obtains the same acoustic wave equations describing compression waves in a fluid or a gas. Note that, historically, the sign conventions for pressure and stress are different in fluid and solid mechanics.

**Visco-acoustic waves.** Generalized Standard Linear Solids can be reduced to acoustics. The corresponding retarded material law for the hydrostatic pressure takes the form

$$\partial_t p(t) = \kappa\nabla\cdot\mathbf{v}(t) + \int_0^t \dot\kappa(t-s)\nabla\cdot\mathbf{v}(s)\,\mathrm{d}s\,, \qquad \dot\kappa(s) = -\sum_{j=1}^r \frac{\kappa_j}{\tau_j}\exp\Big(-\frac{s}{\tau_j}\Big)\,.$$

Defining $\kappa = \kappa_0 + \kappa_1 + \cdots + \kappa_r$ and $p = p_0 + p_1 + \cdots + p_r$ with

$$p_j(t) = \int_0^t \exp\left(\frac{s-t}{\tau_j}\right)\kappa_j\nabla\cdot\mathbf{v}(s)\,\mathrm{d}s\,, \qquad j = 1,\dots,r$$

results in the first-order system for linear visco-acoustic waves

$$\begin{aligned} \rho\,\partial_t\mathbf{v} - \nabla(p_0 + \cdots + p_r) &= \mathbf{f}\,,\\ \partial_t p_0 - \kappa_0\nabla\cdot\mathbf{v} &= 0\,,\\ \partial_t p_j - \kappa_j\nabla\cdot\mathbf{v} + \tau_j^{-1}p_j &= 0\,, \qquad j = 1,\dots,r\,. \end{aligned}$$

This is complemented by initial and boundary conditions (14c)–(14f).

## 1.7. Electro-magnetic waves

Electric fields induce magnetic fields and vice versa. This is formulated by Maxwell's equations describing the propagation of electro-magnetic waves.

**Configuration.** We consider a spatial domain $\Omega \subset \mathbb{R}^3$, a time interval $I = (0, T)$, and a boundary decomposition $\partial\Omega = \Gamma_{\mathrm{E}} \cup \Gamma_{\mathrm{I}}$ corresponding to perfect conducting or transmission boundaries.

**Constituents.** Electro-magnetic waves are determined by the *electric field* and the *magnetic field intensity*

$$\mathbf{E} \colon \overline{I \times \Omega} \to \mathbb{R}^3 \,, \qquad \mathbf{H} \colon \overline{I \times \Omega} \to \mathbb{R}^3 \,,$$

and by the *electric flux density* and *magnetic induction*

$$\mathbf{D} \colon \overline{I \times \Omega} \to \mathbb{R}^3 \,, \qquad \mathbf{B} \colon \overline{I \times \Omega} \to \mathbb{R}^3 \,.$$

Further quantities are the *electric current density* and the *electric charge density*

$$\mathbf{J} \colon I \times \Omega \to \mathbb{R}^3 \,, \qquad \rho \colon I \times \Omega \to \mathbb{R} \,.$$

**Balance relations.** Faraday's law states that the temporal change of the magnetic induction through a two-dimensional subset $A \subset \Omega$ induces an electric field along the boundary $\partial A$, so that for all $0 < t_1 < t_2 < T$

$$\int_A \big(\mathbf{B}(t_2) - \mathbf{B}(t_1)\big) \cdot \mathrm{d}\mathbf{a} = -\int_{t_1}^{t_2} \int_{\partial A} \mathbf{E} \cdot \mathrm{d}\ell \; \mathrm{d}t \,.$$

Ampere's law states that the temporal change of the electric flux density together with the electric current density through a two-dimensional manifold $A \subset \Omega$ induces a magnetic field intensity along the boundary $\partial A$, i.e.,

$$\int_A \big(\mathbf{D}(t_2) - \mathbf{D}(t_1)\big) \cdot \mathrm{d}\mathbf{a} + \int_{t_1}^{t_2} \int_A \mathbf{J} \cdot \mathrm{d}\mathbf{a} \; \mathrm{d}t = \int_{t_1}^{t_2} \int_{\partial A} \mathbf{H} \cdot \mathrm{d}\ell \; \mathrm{d}t \,.$$

Here, we use $\mathbf{u} \cdot \mathrm{d}\mathbf{a} = \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}a$ and $\mathbf{u} \cdot \mathrm{d}\ell = \mathbf{u} \cdot \boldsymbol{\tau} \, \mathrm{d}\ell$, the normal vector field $\mathbf{n} \colon A \to \mathbb{R}^3$ and the tangential vector field $\boldsymbol{\tau} \colon \partial A \to \mathbb{R}^3$ (where the orientation of $\partial A$ is given by $\mathbf{n}$).
The Gauß laws state for all subvolumes $K \subset \Omega$ the conservation of the magnetic induction

$$\int_{\partial K} \mathbf{B} \cdot \mathrm{d}\mathbf{a} = 0$$

and the equilibrium of electric charge density in the volume with electric flux density across the boundary $\partial K$

$$\int_{\partial K} \mathbf{D} \cdot \mathrm{d}\mathbf{a} = \int_K \rho \; \mathrm{d}\mathbf{x} \,.$$

Together, by the integral theorems of Stokes and Gauß we obtain

$$\begin{aligned}
\int_A \int_{t_1}^{t_2} \partial_t \mathbf{B} \cdot \mathrm{d}\mathbf{a} \; \mathrm{d}t &= -\int_{t_1}^{t_2} \int_A \nabla \times \mathbf{E} \cdot \mathrm{d}\mathbf{a} \; \mathrm{d}t \,, & \int_K \nabla \cdot \mathbf{B} \; \mathrm{d}\mathbf{x} &= 0 \,, \\
\int_A \int_{t_1}^{t_2} \partial_t \mathbf{D} \cdot \mathrm{d}\mathbf{a} \; \mathrm{d}t + \int_{t_1}^{t_2} \int_A \mathbf{J} \cdot \mathrm{d}\mathbf{a} \; \mathrm{d}t &= \int_{t_1}^{t_2} \int_A \nabla \times \mathbf{H} \cdot \mathrm{d}\mathbf{a} \; \mathrm{d}t, & \int_K \nabla \cdot \mathbf{D} \; \mathrm{d}\mathbf{x} &= \int_K \rho \; \mathrm{d}\mathbf{x} \,,
\end{aligned}$$

and since this holds for all $(t_1, t_2) \subset I$ and all $A, K \subset \Omega$, it results in the Maxwell system

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = \mathbf{0} \,, \quad \partial_t \mathbf{D} - \nabla \times \mathbf{H} = -\mathbf{J} \,, \quad \nabla \cdot \mathbf{B} = 0 \,, \quad \nabla \cdot \mathbf{D} = \rho \,. \tag{15}$$

Note that a combination of the second and fourth equation implies the conservation of charge $\partial_t \rho + \nabla \cdot \mathbf{J} = 0$.

**Material laws in vacuum.** Without the interaction with matter, electric field and the electric flux density, $\mathbf{D} = \varepsilon_0 \mathbf{E}$, and magnetic induction and magnetic field intensity, $\mathbf{B} = \mu_0 \mathbf{H}$, are proportional by multiplication with the constant *permittivity* $\varepsilon_0$ and *permeability* $\mu_0$, respectively, which together results in the linear second-order Maxwell equation for $\mathbf{E}$

$$\partial_t^2 \mathbf{E} - c^2 \nabla \times \nabla \times \mathbf{E} = \mathbf{0}$$

with speed of light $c = 1/\sqrt{\varepsilon_0 \mu_0}$. A corresponding equation holds for $\mathbf{H}$. In vacuum, in the absence of electric currents and electric charges, we find $\mathbf{J} = \mathbf{0}$ and $\rho = 0$.

**Effective material laws for electro-magnetic waves in matter.** The interaction of electro-magnetic waves with the atoms in matter are described by the *polarization* $\mathbf{P}$ and the *magnetization* $\mathbf{M}$ depending on the electric field $\mathbf{E}$ and the magnetic induction $\mathbf{B}$. For the electric flux density holds

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}(\mathbf{E}, \mathbf{B}) \,,$$

and the magnetic field intensity is given by

$$\mu_0 \mathbf{H} = \mathbf{B} - \mathbf{M}(\mathbf{E}, \mathbf{B}) \,.$$

The electric current density depends on the conductivity $\sigma$ (*Ohm's law*) and the external current $\mathbf{J}_0$, so that

$$\mathbf{J} = \sigma \mathbf{E} + \mathbf{J}_0 \,.$$

In case of linear materials with instantaneous response, the polarization is proportional to the electric field

$$\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$$

with the *susceptibility* $\chi$, that yields $\mathbf{D} = \varepsilon_{\mathrm{r}} \mathbf{E}$ with relative permittivity $\varepsilon_{\mathrm{r}} = \varepsilon_0 (1 + \chi)$.
Linear materials with retarded response are given by

$$\mathbf{P}(t) = \varepsilon_0 \int_{-\infty}^{t} \chi(t - s) \mathbf{E}(s) \, \mathrm{d}s \,. \tag{16}$$

A special case is the Debye model with $\chi(t) = \exp\Big(-\dfrac{t}{\tau}\Big) \dfrac{\varepsilon_{\mathrm{s}} - \varepsilon_\infty}{\tau}$, so that the polarization is determined by

$$\tau \partial_t \mathbf{P} + \mathbf{P} = \varepsilon_0 (\varepsilon_{\mathrm{s}} - \varepsilon_\infty) \mathbf{E} \,.$$

This model is dispersive with a dispersion relation similar to the model for viscous elastic waves.
The relation (16) extends to nonlinear materials by, e.g.,

$$\mathbf{P}(t) = \varepsilon_0 \int_{-\infty}^{t} \chi_1(t - s) \mathbf{E}(s) \, \mathrm{d}s + \int_{-\infty}^{t} \int_{-\infty}^{t} \int_{-\infty}^{t} \chi_3(t - s_1, t - s_2, t - s_3) \big(\mathbf{E}(s_1), \mathbf{E}(s_2), \mathbf{E}(s_3)\big) \, \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 \,.$$

For materials of Kerr-type this response is instantaneous, i.e.,

$$\mathbf{P} = \chi_1 \mathbf{E} + \chi_3 |\mathbf{E}|^2 \mathbf{E} \,.$$

In more complex material models, the Maxwell system (15) is coupled to evolution equations for polarization or magnetization. E.g., in the Maxwell–Lorentz system the evolution of the polarization is determined by

$$\partial_t^2 \mathbf{P} = \frac{1}{\varepsilon_0^2} (\mathbf{E} - \mathbf{P}) + |\mathbf{P}|^2 \mathbf{P} \,.$$

In the Landau–Lifshitz–Gilbert (LLG) equation the magnetization $\mathbf{M}$ is given by

$$\partial_t \mathbf{M} - \alpha \mathbf{M} \times \partial_t \mathbf{M} = -\mathbf{M} \times \mathbf{H}_{\mathrm{eff}} \,, \qquad |\mathbf{M}| = 1 \,,$$

where $\alpha > 0$ is a damping factor, and the effective field $\mathbf{H}_{\mathrm{eff}}$ is a combination of the external magnetic field and the demagnetizing field, which is a magnetic field due to the magnetization.

**Boundary conditions.** The Maxwell system is complemented by conditions on $\partial\Omega$. On a perfectly conducting boundary $\Gamma_{\mathrm{E}}$, we have

$$\mathbf{E} \times \mathbf{n} = \mathbf{0} \text{ and } \mathbf{B} \cdot \mathbf{n} = 0\,,$$

and on the impedance (or Silver–Müller) boundary $\Gamma_{\mathrm{I}}$, we prescribe

$$\mathbf{H} \times \mathbf{n} + \zeta\,(\mathbf{E} \times \mathbf{n}) \times \mathbf{n} = \mathbf{0}$$

depending on the given impedance $\zeta$.
Together, we obtain for general nonlinear instantaneous material laws $\mathbf{D}(\mathbf{E},\mathbf{H})$ and $\mathbf{B}(\mathbf{E},\mathbf{H})$ the first-order system

$$\begin{aligned}
\partial_t\mathbf{D}(\mathbf{E},\mathbf{H}) - \nabla\times\mathbf{H} + \sigma\mathbf{E} &= -\mathbf{J}_0, && \text{in } (0,T)\times\Omega\,, && \text{(17a)}\\
\partial_t\mathbf{B}(\mathbf{E},\mathbf{H}) + \nabla\times\mathbf{E} &= \mathbf{0} && \text{in } (0,T)\times\Omega\,, && \text{(17b)}\\
\mathbf{E}(0) &= \mathbf{E}_0 && \text{in } \Omega \text{ at } t=0\,, && \text{(17c)}\\
\mathbf{H}(0) &= \mathbf{H}_0 && \text{in } \Omega \text{ at } t=0\,, && \text{(17d)}\\
\mathbf{E}\times\mathbf{n} &= \mathbf{0} && \text{on } \Gamma_{\mathrm{E}} \text{ for } t\in(0,T)\,, && \text{(17e)}\\
\mathbf{H}\times\mathbf{n} + \zeta\;(\mathbf{E}\times\mathbf{n})\times\mathbf{n} &= \mathbf{g} && \text{on } \Gamma_{\mathrm{I}} \text{ for } t\in(0,T)\,. && \text{(17f)}
\end{aligned}$$

In nonlinear optics, for the special case of an instantaneous nonmagnetic material law $\mathbf{D}(\mathbf{E}) = \varepsilon_0\mathbf{E} + \mathbf{P}(\mathbf{E})$ and $\mathbf{M} \equiv \mathbf{0}$, the Maxwell system reduces to the second-order equation

$$\partial_t^2\mathbf{D}(\mathbf{E}) + \mu_0^{-1}\nabla\times\nabla\times\mathbf{E} + \sigma\mathbf{E} = -\partial_t\mathbf{J}_0$$

complemented by initial and boundary conditions.

**Bibliographic comments.** The mathematical foundations of modeling elastic solids (including a detailed discussion and a proof of the Cauchy theorem) is given in [Ciarlet, 1988], and more physical background is given in [Davis, 2012]. For generalized standard linear solids we refer to [Fichtner, 2011]. An overview on modeling of electro-magnetic waves is given in [Jackson, 1999], the mathematical aspects of photonics are considered in [Dörfler et al., 2011]. The example (3) is taken from [Leis, 2013, Example 3.4]. Dispersion relations and the analogy in the modeling of elastic and electro-magnetic waves are collected in [Carcione, 2014, Chap. 2 and Chap. 8].

## 2. Space-time solutions for linear hyperbolic systems

The linear wave equation can be analyzed in the framework of symmetric Friedrichs systems as a special case of linear hyperbolic conservation laws. Here, we introduce a general framework for the existence and uniqueness of strong and weak solutions in space and time which applies to general linear wave equations.

We consider operators in space and time of the form $L = M\partial_t + A$ describing a linear hyperbolic system, where $A$ is a first-order operator in space. All results transfer to operators of the form $L = M\partial_t + A + D$ with an additional positive semi-definite operator $D$; this applies to visco-acoustic and visco-elastic models, to mixed boundary conditions of Robin type and impedance boundary conditions.

In the following, we use standard notations: for open domains $G \subset \mathbb{R}^d$ in space or $G \subset \mathbb{R}^{1+d}$ in space-time and functions $v, w \colon G \to \mathbb{R}$ we define the inner product $(v,w)_G = \int_G vw \,\mathrm{d}\mathbf{x}$, the norm $\|v\|_G = \sqrt{(v,v)_G}$ and the Hilbert space $\mathrm{L}_2(G)$ of measurable functions $v \colon G \to \mathbb{R}$ with $\|v\|_G < \infty$.

### 2.1. Linear hyperbolic first-order systems

Let $\Omega \subset \mathbb{R}^d$ be a domain in space with Lipschitz boundary, $I = (0,T)$ a time interval, and we denote the space-time cylinder by $Q = (0,T) \times \Omega$. Boundary conditions will be imposed on $\Gamma_k \subset \partial\Omega$ for $k = 1, \dots, m$, depending on the model, so that the corresponding equations are well-posed.

We consider a linear operator in space and time of the form $L = M\partial_t + A$ with a uniformly positive definite operator $M$ defined by $M\mathbf{y}(\mathbf{x}) = \underline{M}(\mathbf{x})\mathbf{y}(\mathbf{x})$ with a matrix valued function $\underline{M} \in \mathrm{L}_\infty(\Omega; \mathbb{R}^{m\times m}_{\mathrm{sym}})$, and a differential operator $A\mathbf{y} = \sum_{j=1}^d \underline{A}_j \partial_j \mathbf{y}$ with matrices $\underline{A}_j \in \mathbb{R}^{m\times m}_{\mathrm{sym}}$. Moreover, we define the matrix $\underline{A}_{\mathbf{n}} = \sum_{j=1}^d n_j \underline{A}_j \in \mathbb{R}^{m\times m}_{\mathrm{sym}}$ for $\mathbf{n} \in \mathbb{R}^d$ and the corresponding boundary operator $(A_{\mathbf{n}}\mathbf{y})(\mathbf{x}) = \underline{A}_{\mathbf{n}}\mathbf{y}(\mathbf{x})$.

In the first step, we consider the properties of the operators $A$ and $L$ for smooth functions. Then the operators are extended to Hilbert spaces and, by specifying boundary conditions, we define maximal domains for the operators.

**Example 4.** This applies to the linear acoustic wave equation (14) with $m = d + 1$ and

$$\mathbf{y} = \begin{pmatrix} \mathbf{v} \\ p \end{pmatrix}, \quad M\mathbf{y} = \begin{pmatrix} \rho\mathbf{v} \\ \kappa^{-1}p \end{pmatrix}, \quad A\mathbf{y} = \begin{pmatrix} -\nabla p \\ -\nabla\cdot\mathbf{v} \end{pmatrix}, \quad \underline{A}_{\mathbf{n}}\mathbf{y} = \begin{pmatrix} -p\mathbf{n} \\ -\mathbf{n}\cdot\mathbf{v} \end{pmatrix}. \tag{18}$$

For linear elastic waves with $\mathbf{v} = \partial_t\mathbf{u}$ and $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u})$ we have

$$\mathbf{y} = \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\sigma} \end{pmatrix}, \; M\mathbf{y} = \begin{pmatrix} \rho\mathbf{v} \\ \mathbf{C}^{-1}\boldsymbol{\sigma} \end{pmatrix}, \; A\mathbf{y} = \begin{pmatrix} -\operatorname{div}\boldsymbol{\sigma} \\ -\boldsymbol{\varepsilon}(\mathbf{v}) \end{pmatrix}, \; \underline{A}_{\mathbf{n}}\mathbf{y} = \begin{pmatrix} -\boldsymbol{\sigma}\mathbf{n} \\ -\frac{1}{2}(\mathbf{n}\mathbf{v}^\top + \mathbf{v}\mathbf{n}^\top) \end{pmatrix}, \tag{19}$$

and $\frac{1}{2}M\mathbf{y}\cdot\mathbf{y} = \frac{1}{2}\big(\rho|\mathbf{v}|^2 + \boldsymbol{\sigma}\cdot\mathbf{C}^{-1}\boldsymbol{\sigma}\big) = \frac{1}{2}\big(\rho|\partial_t\mathbf{u}|^2 + \boldsymbol{\varepsilon}(\mathbf{u})\cdot\mathbf{C}\boldsymbol{\varepsilon}(\mathbf{u})\big)$ is the kinetic and potential energy.

For linear electro-magnetic waves we have

$$\mathbf{y} = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \quad M\mathbf{y} = \begin{pmatrix} \varepsilon_0\mathbf{E} \\ \mu_0\mathbf{H} \end{pmatrix}, \quad A\mathbf{y} = \begin{pmatrix} -\nabla\times\mathbf{H} \\ \nabla\times\mathbf{E} \end{pmatrix}, \quad \underline{A}_{\mathbf{n}}\mathbf{y} = \begin{pmatrix} -\mathbf{n}\times\mathbf{H} \\ \mathbf{n}\times\mathbf{E} \end{pmatrix}, \tag{20}$$

and $\frac{1}{2}M\mathbf{y}\cdot\mathbf{y} = \frac{1}{2}\big(\varepsilon_0|\mathbf{E}|^2 + \mu_0|\mathbf{H}|^2\big)$ is the electro-magnetic energy.

**Linear conservation laws.** Defining

$$\underline{A} = (\underline{A}_1, \cdots, \underline{A}_d) = (A_{j,kl})_{j=1\dots d,\ k,l=1,\dots,m} \in \mathbb{R}^{d\times m\times m}$$

we observe $A\mathbf{y} = \sum_{j=1}^d \partial_j \underline{A}_j \mathbf{y} = \operatorname{div}(\underline{A}\mathbf{y})$ and $\underline{A}_{\mathbf{n}} = \mathbf{n}\cdot\underline{A} = \sum\limits_{j=1}^{d} n_j \underline{A}_j$, so that this system takes the form of a *linear conservation law*

$$\underline{M}\partial_t \mathbf{y} + \operatorname{div}(\underline{A}\mathbf{y}) = \mathbf{f}\,.$$

Integration by parts and using the symmetry of $\underline{A}_j$ yields for differentiable functions with compact support in $\Omega$

$$\begin{aligned}
(A\mathbf{y},\mathbf{z})_\Omega &= \sum_{j=1}^d \int_\Omega \underline{A}_j \partial_j \mathbf{y}\cdot\mathbf{z}\,\mathrm{d}\mathbf{x} = \sum_{j=1}^d \sum_{k,l=1}^m \int_\Omega A_{j,kl}(\partial_j y_l) z_k\,\mathrm{d}\mathbf{x}\\
&= -\sum_{j=1}^d \sum_{k,l=1}^m \int_\Omega A_{j,kl} y_l\,\partial_j z_k\,\mathrm{d}\mathbf{x} = -\sum_{j=1}^d \sum_{k,l=1}^m \int_\Omega y_l A_{j,lk}\,\partial_j z_k\,\mathrm{d}\mathbf{x}\\
&= -\sum_{j=1}^d \int_\Omega \mathbf{y}\cdot\underline{A}_j\partial_j\mathbf{z}\,\mathrm{d}\mathbf{x} = -(\mathbf{y}, A\mathbf{z})_\Omega\,, \qquad \mathbf{y},\mathbf{z}\in \mathrm{C}^1_\mathrm{c}(\Omega;\mathbb{R}^m)\,,
\end{aligned}$$

so that $A^* = -A$ on $\mathrm{C}^1_\mathrm{c}(\Omega;\mathbb{R}^m)$. On the boundary $\partial\Omega$ with outer unit normal $\mathbf{n}$, integration by parts yields

$$\begin{aligned}
(A\mathbf{y},\mathbf{z})_\Omega + (\mathbf{y}, A\mathbf{z})_\Omega &= \sum_{j=1}^d \sum_{k,l=1}^m \int_\Omega \Big(A_{j,kl}(\partial_j y_l) z_k + y_l A_{j,lk}\,\partial_j z_k\Big)\,\mathrm{d}\mathbf{x}\\
&= \sum_{j=1}^d \sum_{k,l=1}^m \int_\Omega \partial_j\big(A_{j,kl} y_l z_k\big)\,\mathrm{d}\mathbf{x} = \sum_{j=1}^d \sum_{k,l=1}^m \int_\Omega n_j\, A_{j,kl} y_l z_k\,\mathrm{d}\mathbf{a}\\
&= \int_{\partial\Omega} \underline{A}_{\mathbf{n}}\mathbf{y}\cdot\mathbf{z}\,\mathrm{d}\mathbf{a} = (\underline{A}_{\mathbf{n}}\mathbf{y},\mathbf{z})_{\partial\Omega}\,, \qquad \mathbf{y},\mathbf{z}\in\mathrm{C}^1(\Omega;\mathbb{R}^m)\cap\mathrm{C}^0(\overline{\Omega};\mathbb{R}^m)\,.
\end{aligned}$$

Together, we obtain in space and time for $L = M\partial_t + A$ and its adjoint $L^* = -L$

$$(L\mathbf{v},\mathbf{w})_Q - (\mathbf{v}, L^*\mathbf{w})_Q = \big(M\mathbf{v}(T),\mathbf{w}(T)\big)_\Omega - \big(M\mathbf{v}(0),\mathbf{w}(0)\big)_\Omega + (\underline{A}_{\mathbf{n}}\mathbf{v},\mathbf{w})_{(0,T)\times\partial\Omega} \tag{21}$$

for $\mathbf{v},\mathbf{w}\in\mathrm{C}^1(Q;\mathbb{R}^m)\cap\mathrm{C}^0(\overline{Q};\mathbb{R}^m)$.

**Example 5.** For linear acoustic waves (18) we have

$$\begin{aligned}
\big(L(\mathbf{v},p),(\mathbf{w},q)\big)_Q + \big((\mathbf{v},p),L(\mathbf{w},q)\big)_Q = {}& \big(\rho\mathbf{v}(T),\mathbf{w}(T)\big)_\Omega + \big(\kappa^{-1}p(T),p(T)\big)_\Omega\\
&- \big(\rho\mathbf{v}(0),\mathbf{w}(0)\big)_\Omega - \big(\kappa^{-1}p(0),p(0)\big)_\Omega\\
&- (p,\mathbf{n}\cdot\mathbf{w})_{(0,T)\times\partial\Omega} - (\mathbf{n}\cdot\mathbf{v},q)_{(0,T)\times\partial\Omega}\,.
\end{aligned}$$

For linear elastic waves (19) we have

$$\begin{aligned}
\big(L(\mathbf{v},\boldsymbol{\sigma}),(\mathbf{w},\boldsymbol{\tau})\big)_Q + \big((\mathbf{v},\boldsymbol{\sigma}),L(\mathbf{w},\boldsymbol{\tau})\big)_Q = {}& \big(\rho\mathbf{v}(T),\mathbf{w}(T)\big)_\Omega + \big(\mathbf{C}^{-1}\boldsymbol{\sigma}(T),\boldsymbol{\tau}(T)\big)_\Omega\\
&- \big(\rho\mathbf{v}(0),\mathbf{w}(0)\big)_\Omega - \big(\mathbf{C}^{-1}\boldsymbol{\sigma}(0),\boldsymbol{\tau}(0)\big)_\Omega\\
&- (\boldsymbol{\sigma}\mathbf{n},\mathbf{w})_{(0,T)\times\partial\Omega} - (\mathbf{v},\boldsymbol{\tau}\mathbf{n})_{(0,T)\times\partial\Omega}\,.
\end{aligned}$$

For linear electro-magnetic waves (20) we have

$$\begin{aligned}
\big(L(\mathbf{E},\mathbf{H}),(\mathbf{e},\mathbf{h})\big)_Q + \big((\mathbf{E},\mathbf{H}),L(\mathbf{e},\mathbf{h})\big)_Q = {}& \big(\varepsilon_0\mathbf{E}(T),\mathbf{e}(T)\big)_\Omega + \big(\mu_0\mathbf{H}(T),\mathbf{h}(T)\big)_\Omega\\
&- \big(\varepsilon_0\mathbf{E}(0),\mathbf{e}(0)\big)_\Omega - \big(\mu_0\mathbf{H}(0),\mathbf{h}(0)\big)_\Omega\\
&- (\mathbf{E}\times\mathbf{n},\mathbf{h})_{(0,T)\times\partial\Omega} + (\mathbf{H}\times\mathbf{n},\mathbf{e})_{(0,T)\times\partial\Omega}\,.
\end{aligned}$$

Here we use the following calculus: for vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ we have $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$, and for vector fields $\mathbf{u}, \mathbf{v} \colon \Omega \to \mathbb{R}^3$ we have $\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\nabla \times \mathbf{u}) - \mathbf{u} \cdot (\nabla \times \mathbf{v})$. Thus, the Gauß theorem gives

$$\int_\Omega \mathbf{v} \cdot (\nabla \times \mathbf{u}) \, \mathrm{d}\mathbf{x} - \int_\Omega \mathbf{u} \cdot (\nabla \times \mathbf{v}) \, \mathrm{d}\mathbf{x} = \int_\Omega \nabla \cdot (\mathbf{u} \times \mathbf{v}) \, \mathrm{d}\mathbf{x} = \int_{\partial\Omega} (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{n} \, \mathrm{d}\mathbf{a} = \int_{\partial\Omega} \mathbf{u} \cdot (\mathbf{v} \times \mathbf{n}) \, \mathrm{d}\mathbf{a} \, .$$

The formulation in our examples of wave equations as Friedrichs systems yields symmetric matrices of the form $\underline{A}_j = \begin{pmatrix} \underline{0} & \underline{\tilde{A}}_j \\ \underline{\tilde{A}}_j^\top & \underline{0} \end{pmatrix}$ with $\underline{\tilde{A}}_j \in \mathbb{R}^{m_1 \times m_2}$ and $m = m_1 + m_2$. In order to obtain a well-posed problem with a unique solution, boundary conditions are required. Here we select $\Gamma_1 = \ldots = \Gamma_{m_1} \subset \partial\Omega$ and the complement $\Gamma_k = \partial\Omega \setminus \overline{\Gamma}_1$ for $k = m_1 + 1, \ldots, m$, as it is specified in the next section for acoustics in Ex. 6.

## 2.2. Solution spaces

We define the Hilbert spaces

$$\begin{aligned}
\mathrm{H}(A,\Omega) &= \big\{\mathbf{y}\in\mathrm{L}_2(\Omega;\mathbb{R}^m)\colon \mathbf{z}\in\mathrm{L}_2(\Omega;\mathbb{R}^m) \text{ exists with } (\mathbf{z},\mathbf{w})_\Omega = (\mathbf{y},A^*\mathbf{w})_\Omega \text{ for all } \mathbf{w}\in\mathrm{C}^1_\mathrm{c}(\Omega;\mathbb{R}^m)\big\}\,,\\
\mathrm{H}(L,Q) &= \big\{\mathbf{v}\in\mathrm{L}_2(Q;\mathbb{R}^m)\colon \mathbf{z}\in\mathrm{L}_2(Q;\mathbb{R}^m) \text{ exists with } (\mathbf{z},\mathbf{w})_Q = (\mathbf{v},L^*\mathbf{w})_Q \text{ for all } \mathbf{w}\in\mathrm{C}^1_\mathrm{c}(Q;\mathbb{R}^m)\big\}\,,
\end{aligned}$$

so that for $\mathbf{y}\in\mathrm{H}(A,\Omega)$ and $\mathbf{v}\in\mathrm{H}(L,Q)$ the weak derivatives $A\mathbf{y}\in\mathrm{L}_2(\Omega;\mathbb{R}^m)$ and $L\mathbf{v}\in\mathrm{L}_2(Q;\mathbb{R}^m)$ exist; the corresponding norms are

$$\|\mathbf{y}\|_{\mathrm{H}(A,\Omega)} = \sqrt{\|\mathbf{y}\|_\Omega^2 + \|A\mathbf{y}\|_\Omega^2}\,,\qquad \|\mathbf{v}\|_{\mathrm{H}(L,Q)} = \sqrt{\|\mathbf{v}\|_Q^2 + \|L\mathbf{v}\|_Q^2}\,.$$

Depending on homogeneous boundary conditions on $\Gamma_k\subset\partial\Omega$, $k=1,\dots,m$, we define

$$\mathcal{Z} = \big\{\mathbf{w}\in\mathrm{C}^1(\Omega;\mathbb{R}^m)\cap\mathrm{C}^0(\overline{\Omega};\mathbb{R}^m)\colon (\underline{A}_\mathbf{n}\mathbf{w})_k = 0 \text{ on } \Gamma_k\,, k=1,\dots,m\big\}, \tag{22a}$$

$$\mathcal{V} = \big\{\mathbf{w}\in\mathrm{C}^1(Q;\mathbb{R}^m)\cap\mathrm{C}^0(\overline{Q};\mathbb{R}^m)\colon \mathbf{w}(0)=\mathbf{0}\,,\qquad (\underline{A}_\mathbf{n}\mathbf{w})_k = 0 \text{ on } (0,T)\times\Gamma_k\,,\ k=1,\dots,m\big\}\,, \tag{22b}$$

$$\mathcal{V}^* = \big\{\mathbf{z}\in\mathrm{C}^1(Q;\mathbb{R}^m)\cap\mathrm{C}^0(\overline{Q};\mathbb{R}^m)\colon \mathbf{z}(T)=\mathbf{0}\,,\qquad (\underline{A}_\mathbf{n}\mathbf{z})_k = 0 \text{ on } (0,T)\times\Gamma_k^*\,,\ k=1,\dots,m\big\}\,, \tag{22c}$$

where the sets $\Gamma_k\subset\partial\Omega$ are chosen such that

$$(A\mathbf{z},\mathbf{z})_\Omega = 0\,,\qquad \mathbf{z}\in\mathcal{Z}\,, \tag{23}$$

and such that for the sets $\Gamma_k^*\subset\partial\Omega$ in the definition of the test space holds

$$(\underline{A}_\mathbf{n}\mathbf{w},\mathbf{z})_{(0,T)\times\partial\Omega} = \sum_{k=1}^m \big((\underline{A}_\mathbf{n}\mathbf{w})_k, z_k\big)_{(0,T)\times\Gamma_k}\,,\quad \mathbf{w}\in\mathrm{C}^1(Q;\mathbb{R}^m)\,,\ \mathbf{z}\in\mathcal{V}^*\,. \tag{24}$$

This is obtained by taking $\Gamma_k^*\subset\partial\Omega$ minimal such that for homogeneous boundary conditions in $\mathcal{V}$ and $\mathcal{V}^*$

$$(\underline{A}_\mathbf{n}\mathbf{w},\mathbf{z})_{(0,T)\times\partial\Omega} = 0\,,\qquad \mathbf{w}\in\mathcal{V}\,,\ \mathbf{z}\in\mathcal{V}^* \tag{25}$$

The choice of $\Gamma_k$ and $\Gamma_k^*$ is essential in order to obtain a well-posed problem; this will be explained for our examples in Sec. 2.7. Since we have $A^*=-A$, this implies $(A\mathbf{z},\mathbf{z})_\Omega = \frac12(\underline{A}_\mathbf{n}\mathbf{z},\mathbf{z})_{\partial\Omega}$, and we observe $\Gamma_k^*=\Gamma_k$. Note that this is specific for our applications to wave problems but does not apply to general linear hyperbolic systems.
Let $Z\subset\mathrm{H}(A,Q)$ be the closure of $\mathcal{Z}$ with respect to the norm $\|\cdot\|_{\mathrm{H}(A,Q)}$, let $V\subset\mathrm{H}(L,Q)$ be the closure of $\mathcal{V}$ with respect to the norm $\|\cdot\|_{\mathrm{H}(L,Q)}$, and let $V^*\subset\mathrm{H}(L^*,Q)$ be the closure of $\mathcal{V}^*$ with respect to the norm $\|\cdot\|_{\mathrm{H}(L^*,Q)}$. Then, we obtain from (21) and (24)

$$(L\mathbf{v},\mathbf{w})_Q - (\mathbf{v},L^*\mathbf{w})_Q = 0\,,\qquad \mathbf{v}\in V\,,\ \mathbf{w}\in V^*\,. \tag{26}$$

**Example 6.** For linear acoustic waves (18) we have $\mathrm{H}(A,\Omega)=\mathrm{H}(\mathrm{div},\Omega)\times\mathrm{H}^1(\Omega)$, and for $d=2$ the boundary parts $\Gamma_1=\Gamma_2=\Gamma_\mathrm{S}$ and $\Gamma_3=\Gamma_\mathrm{V}$ with $\partial\Omega=\Gamma_\mathrm{S}\cup\Gamma_\mathrm{V}$ in Ex. 5 yields that (26) holds with $\Gamma_k=\Gamma_k^*$, and we obtain

$$\begin{aligned}
Z &= \big\{(\mathbf{v},p)\in\mathrm{H}(\mathrm{div},\Omega)\times\mathrm{H}^1(\Omega)\colon \mathbf{v}\cdot\mathbf{n}=0 \text{ on } \Gamma_\mathrm{V}\,,\ p=0 \text{ on } \Gamma_\mathrm{S}\big\}\,,\\
V &\supset \big\{(\mathbf{v},p)\in\mathrm{H}^1(0,T;\mathrm{L}_2(\Omega;\mathbb{R}^m))\cap\mathrm{L}_2(0,T;\mathrm{H}(\mathrm{div},\Omega)\times\mathrm{H}^1(\Omega))\colon\\
&\qquad \mathbf{v}(0)=\mathbf{0}\,,\ p(0)=0\,,\mathbf{v}\cdot\mathbf{n}=0 \text{ on } (0,T)\times\Gamma_\mathrm{V}\,,\ p=0 \text{ on } (0,T)\times\Gamma_\mathrm{S}\big\}\,,\\
V^* &\supset \big\{(\mathbf{w},q)\in\mathrm{H}^1(0,T;\mathrm{L}_2(\Omega;\mathbb{R}^m))\cap\mathrm{L}_2(0,T;\mathrm{H}(\mathrm{div},\Omega)\times\mathrm{H}^1(\Omega))\colon\\
&\qquad \mathbf{w}(T)=\mathbf{0}\,,\ q(T)=0\,,\mathbf{w}\cdot\mathbf{n}=0 \text{ on } (0,T)\times\Gamma_\mathrm{V}\,,\ q=0 \text{ on } (0,T)\times\Gamma_\mathrm{S}\big\}\,.
\end{aligned}$$

In $Y=\mathrm{L}_2(\Omega;\mathbb{R}^m)$ and $W=\mathrm{L}_2(Q;\mathbb{R}^m)$ we use the energy norms

$$\|\mathbf{y}\|_Y = \sqrt{(M\mathbf{y},\mathbf{y})_\Omega}\,,\quad \mathbf{y}\in Y\,,\qquad \|\mathbf{w}\|_W = \sqrt{(M\mathbf{w},\mathbf{w})_Q}\,,\quad \mathbf{w}\in W\,,$$

and for the $\mathrm{L}_2$ adjoints

$$\|\mathbf{y}\|_{Y^*} = \sup_{\mathbf{z}\in Y\setminus\{\mathbf{0}\}} \frac{(\mathbf{y},\mathbf{z})_\Omega}{\|\mathbf{z}\|_Y} = \sqrt{(M^{-1}\mathbf{y},\mathbf{y})_\Omega}\,, \qquad \|\mathbf{w}\|_{W^*} = \sqrt{(M^{-1}\mathbf{w},\mathbf{w})_Q}\,.$$

In $V$ and $V^*$ we use the weighted norms

$$\|\mathbf{v}\|_V = \sqrt{\|\mathbf{v}\|_W^2 + \|L\mathbf{v}\|_{W^*}^2}\,, \quad \|\mathbf{z}\|_{V^*} = \sqrt{\|\mathbf{z}\|_W^2 + \|L^*\mathbf{z}\|_{W^*}^2}\,, \quad \mathbf{v}\in V\,,\ \mathbf{z}\in V^*\,.$$

**Remark 7.** For the extension to visco-acoustic and visco-elastic models the same solution spaces can be used. For mixed boundary conditions of Robin type or impedance boundary conditions a modification is required to include additional conditions on the boundary, see Rem. 22. This relies on the fact that traces are well-defined for smooth test functions in $\mathcal{V}^*$, but in general not in $V$, where traces on mixed boundaries are only defined in distributional sense.

## 2.3. Solution concepts

We consider different solution spaces of the equation $L\mathbf{u} = \mathbf{f}$ with initial and boundary conditions.

**Definition 8.** *Depending on regularity of the data, we define:*

a) $\mathbf{u} \in \mathrm{C}^1(Q;\mathbb{R}^m) \cap \mathrm{C}^0(\overline{Q};\mathbb{R}^m)$ *is a* classical solution*, if*

$$\begin{aligned} L\mathbf{u} &= \mathbf{f} && \textit{in } Q = (0,T)\times\Omega\,,\\ \mathbf{u}(0) &= \mathbf{u}_0 && \textit{in } \Omega \textit{ at } t = 0\,,\\ (\underline{A}_\mathbf{n}\mathbf{u})_k &= g_k && \textit{on } (0,T)\times\Gamma_k\,,\ k = 1,\dots,m\,, \end{aligned}$$

*for* $\mathbf{f} \in \mathrm{C}^0(Q;\mathbb{R}^m)$, $\mathbf{u}_0 \in \mathrm{C}^0(\Omega;\mathbb{R}^m)$, $g_k \in \mathrm{C}^0((0,T)\times\Gamma_k)$.

b) $\mathbf{u} \in \mathrm{H}(L,Q)$ *is a* strong solution*, if*

$$\begin{aligned} L\mathbf{u} &= \mathbf{f} && \textit{in } Q = (0,T)\times\Omega\,,\\ \mathbf{u}(0) &= \mathbf{u}_0 && \textit{in } \Omega \textit{ at } t = 0\,,\\ (\underline{A}_\mathbf{n}\mathbf{u})_k &= g_k && \textit{on } (0,T)\times\Gamma_k\,,\ k = 1,\dots,m\,, \end{aligned}$$

*for* $\mathbf{f} \in \mathrm{L}_2(Q;\mathbb{R}^m)$, $\mathbf{u}_0 \in \mathrm{L}_2(\Omega;\mathbb{R}^m)$, $g_k \in \mathrm{L}_2((0,T)\times\Gamma_k)$.

c) $\mathbf{u} \in \mathrm{L}_2(Q;\mathbb{R}^m)$ *is a* weak solution*, if*

$$\big(\mathbf{u}, L^*\mathbf{z}\big)_Q = \langle \ell, \mathbf{z}\rangle\,, \qquad \mathbf{z} \in \mathcal{V}^*,$$

*with the linear functional* $\ell$ *defined by*

$$\langle \ell, \mathbf{z}\rangle = (\mathbf{f}, \mathbf{z})_Q + \big(M\mathbf{u}_0, \mathbf{z}(0)\big)_\Omega - (\mathbf{g}, \mathbf{z})_{(0,T)\times\partial\Omega}$$

*for data* $\mathbf{f} \in \mathrm{L}_2(Q;\mathbb{R}^m)$, $\mathbf{u}_0 \in \mathrm{L}_2(\Omega;\mathbb{R}^m)$, *and* $g_k \in \mathrm{L}_2((0,T)\times\Gamma_k)$.
*We set* $\mathbf{g} = (g_k)_{k=1,\dots,m} \in \mathrm{L}_2((0,T)\times\partial\Omega;\mathbb{R}^m)$ *with* $g_k = 0$ *on* $\partial\Omega\setminus\Gamma_k$.

**Remark 9.** For the variational definition of weak solutions we use smooth test functions $\mathcal{V}^*$ so that the space-time traces on $\{0\}\times\Omega \subset \partial Q$ and $(0,T)\times\partial\Omega \subset \partial Q$ are well defined; with additional assumptions in Thm. 11 and Thm. 35 this extends to test functions in $V^*$.

**Example 10.** A weak solution $(v,\sigma) \in \mathrm{L}_2((0,T)\times(0,X);\mathbb{R}^2)$ of the linear wave equation (2) in 1d with wave speed $c = \sqrt{\kappa/\rho}$ and homogeneous Dirichlet boundary conditions satisfies

$$\big(v, -\rho\partial_t w + \partial_x\tau\big)_{(0,T)\times(0,X)} + \big(\sigma, -\kappa^{-1}\partial_t\tau + \partial_x w\big)_{(0,T)\times(0,X)} = \big(v_0, w(0)\big)_{(0,X)} + \big(\sigma_0, \tau(0)\big)_{(0,X)}$$

for all test functions $w,\tau \in \mathrm{C}^1([0,T]\times[0,X])$ with $w(T,x) = \tau(T,x) = 0$ for $x \in (0,X)$ and $w(t,0) = w(t,X) = 0$ for $t \in (0,T)$. This allows for discontinuities of the solution along the characteristics

$$\left\{\begin{pmatrix} t\\ x_0 \pm ct\end{pmatrix} \in (0,T)\times\mathbb{R} : x_0 \pm ct \in \Omega\right\} = \left\{\begin{pmatrix} t\\ x\end{pmatrix} \in (0,T)\times\Omega : \begin{pmatrix} t\\ x - x_0\end{pmatrix}\cdot\begin{pmatrix} \pm c\\ 1\end{pmatrix} = 0\right\}$$

for some $x_0 \in \mathbb{R}$. Here we illustrate this for a simple example: consider a piecewise constant function

$$\begin{pmatrix} v(t,x)\\ \sigma(t,x)\end{pmatrix} = \begin{cases} \begin{pmatrix} v_\mathrm{L}\\ \sigma_\mathrm{L}\end{pmatrix} & \text{for } x < x_0 + ct\,, \qquad [v] = v_\mathrm{R} - v_\mathrm{L}\,,\\ \begin{pmatrix} v_\mathrm{R}\\ \sigma_\mathrm{R}\end{pmatrix} & \text{for } x > x_0 + ct\,, \qquad [\sigma] = \sigma_\mathrm{R} - \sigma_\mathrm{L}\,. \end{cases}$$

Then, we have for all $(w,\tau) \in \mathrm{C_c}([0,T]\times[0,X],\mathbb{R}^2)$

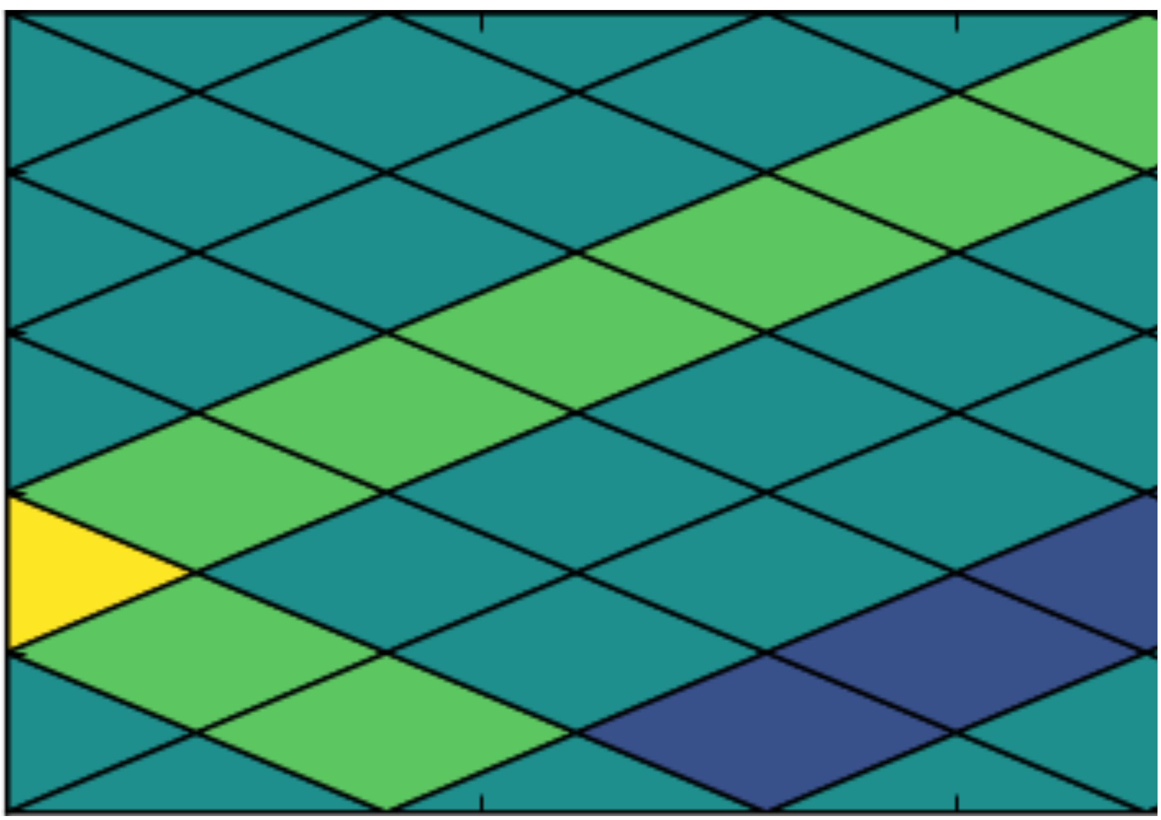
$$\begin{aligned}
\int_0^T \int_0^X & \begin{pmatrix} v \\ \sigma \end{pmatrix} \cdot \begin{pmatrix} -\rho\partial_t w + \partial_x \tau \\ -\kappa^{-1}\partial_t \tau + \partial_x w \end{pmatrix} \mathrm{d}x\ \mathrm{d}t \\
&= \int_{x<x_0+ct} \begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix} \cdot \begin{pmatrix} -\rho v_{\mathrm{L}} w - \kappa^{-1}\sigma_{\mathrm{L}}\tau \\ v_{\mathrm{L}}\tau + \sigma_{\mathrm{L}} w \end{pmatrix} \mathrm{d}x\ \mathrm{d}t + \int_{x>x_0+ct} \begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix} \cdot \begin{pmatrix} -\rho v_{\mathrm{R}} w - \kappa^{-1}\sigma_{\mathrm{R}}\tau \\ v_{\mathrm{R}}\tau + \sigma_{\mathrm{R}} w \end{pmatrix} \mathrm{d}x\ \mathrm{d}t \\
&= \int_{x=x_0+ct} \frac{1}{\sqrt{1+c^2}} \begin{pmatrix} -c \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -\rho v_{\mathrm{L}} w - \kappa^{-1}\sigma_{\mathrm{L}}\tau \\ v_{\mathrm{L}}\tau + \sigma_{\mathrm{L}} w \end{pmatrix} \mathrm{d}a + \int_{x=x_0+ct} \frac{1}{\sqrt{1+c^2}} \begin{pmatrix} c \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -\rho v_{\mathrm{R}} w - \kappa^{-1}\sigma_{\mathrm{R}}\tau \\ v_{\mathrm{R}}\tau + \sigma_{\mathrm{R}} w \end{pmatrix} \mathrm{d}a \\
&= -\frac{1}{\sqrt{1+c^2}} \int_{x=x_0+ct} \begin{pmatrix} c \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -\rho[v]w - \kappa^{-1}[\sigma]\tau \\ [v]\tau + [\sigma]w \end{pmatrix} \mathrm{d}a \\
&= \frac{1}{\sqrt{1+c^2}} \int_{x=x_0+ct} \left( c \begin{pmatrix} \rho & 0 \\ 0 & \kappa^{-1} \end{pmatrix} \begin{pmatrix} [v] \\ [\sigma] \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} [v] \\ [\sigma] \end{pmatrix} \right) \cdot \begin{pmatrix} w \\ \tau \end{pmatrix} \mathrm{d}a \\
&= \frac{1}{\sqrt{1+c^2}} \int_{x=x_0+ct} \left( c\underline{M} \begin{pmatrix} [v] \\ [\sigma] \end{pmatrix} + \underline{A} \begin{pmatrix} [v] \\ [\sigma] \end{pmatrix} \right) \cdot \begin{pmatrix} w \\ \tau \end{pmatrix} \mathrm{d}a\,.
\end{aligned}$$

We observe, that $(v,\sigma)$ is a weak solution if the jump $([v],[\sigma])^\top$ is an eigenvector of

$$\underline{A} \begin{pmatrix} [v] \\ [\sigma] \end{pmatrix} = -c\underline{M} \begin{pmatrix} [v] \\ [\sigma] \end{pmatrix}\,.$$

This is equivalent to the jump conditions $[\sigma] - c\rho[v] = 0$ and $[v] - c\kappa^{-1}[\sigma] = 0$.

FIGURE 2. Illustration of a piecewise constant weak solution in 1d of the wave equation in space and time with jumps along the characteristics. The solution is computed by the explicit time stepping scheme in Example 10.



Based on the jump conditions we construct a weak solution

$$(v,\sigma) \in \mathrm{L}_2\big((0,T)\times(0,X),\mathbb{R}^2\big)$$

with $X = cT$ that is discontinuous along the characteristics $(t, j\triangle x \pm ct)$ on a special mesh in space and time depending of the wave speed $c$ with $\triangle x = c\triangle t$ and $\triangle t = T/N$, $N \in \mathbb{N}$, cf. Fig. 2. Starting with $v(0,x) = v^0_{j-\frac12}$ and $\sigma(0,x) = \sigma^0_{j-\frac12}$ for $(j-1)\triangle x < x < j\triangle x$, we obtain from the jump condition recursively for $n = 1, 2, \dots, N$

$$
\begin{aligned}
v_j^{n-\frac12} &= \frac12\Big(v^{n-1}_{j+\frac12} + v^{n-1}_{j-\frac12} + \sigma^{n-1}_{j+\frac12} - \sigma^{n-1}_{j-\frac12}\Big)\,, & & & v^n_{-\frac12} &= -v^n_{\frac12}\,,\\
\sigma_j^{n-\frac12} &= \frac12\Big(v^{n-\frac12}_{j+\frac12} - v^{n-\frac12}_{j-\frac12} + \sigma^{n-\frac12}_{j+\frac12} + \sigma^{n-\frac12}_{j-\frac12}\Big)\,, & j &= 0,\dots,N\,, & v^n_{N+\frac12} &= -v^n_{N-\frac12}\,,\\
v^n_{j-\frac12} &= \frac12\Big(v_j^{n-\frac12} + v_{j-1}^{n+\frac12} + \sigma_j^{n-\frac12} - \sigma_{j-1}^{n-\frac12}\Big)\,, & & & \sigma^n_{-\frac12} &= \sigma^n_{\frac12}\,,\\
\sigma^n_{j-\frac12} &= \frac12\Big(v_j^{n-\frac12} - v_{j-1}^{n-\frac12} + \sigma_j^{n-\frac12} + \sigma_{j-1}^{n-\frac12}\Big)\,, & j &= 1,\dots,N\,, & \sigma^n_{N+\frac12} &= \sigma^n_{N-\frac12}\,,
\end{aligned}
$$

with suitable extensions for homogeneous Dirichlet boundary conditions for $v$, see Fig. 2 for an example.

### 2.4. **Existence and uniqueness of space-time solutions**

Now we construct strong and weak solutions by a least squares approach. Therefore, we define the quadratic functionals

$$J(\mathbf{v}) = \frac{1}{2}\|L\mathbf{v} - \mathbf{f}\|_{W^*}^2\,, \quad \mathbf{v} \in \mathrm{H}(L, Q)\,, \qquad\qquad J^*(\mathbf{z}) = \frac{1}{2}\|L^*\mathbf{z}\|_{W^*}^2 - \langle \ell, \mathbf{z}\rangle\,, \quad \mathbf{z} \in \mathcal{V}^*\,.$$

**Theorem 11.** *Depending on the regularity of the data, we obtain:*

a) *Assume that $C_L > 0$ exists with*

$$\|\mathbf{v}\|_W \le C_L \|L\mathbf{v}\|_{W^*}\,, \qquad \mathbf{v} \in V\,. \tag{27}$$

*Then, a unique minimizer $\mathbf{u} \in V$ of $J(\cdot)$ exists, and if $L(V) = W$, the minimizer $\mathbf{u} \in V$ is the unique strong solution of*

$$(L\mathbf{u}, \mathbf{w})_Q = (\mathbf{f}, \mathbf{w})_Q\,, \qquad \mathbf{w} \in W \tag{28}$$

*with homogeneous initial and boundary data.*

b) *Assume that $C_{L^*} > 0$ and $C_\ell > 0$ exists with*

$$\|\mathbf{z}\|_W \le C_{L^*}\|L^*\mathbf{z}\|_{W^*}\,, \qquad |\langle \ell, \mathbf{z}\rangle| \le C_\ell \|\mathbf{z}\|_{V^*}\,, \qquad \mathbf{z} \in \mathcal{V}^*\,. \tag{29}$$

*Then, $J^*(\cdot)$ extends to $V^*$, a unique minimizer $\mathbf{z}^* \in V^*$ of $J^*(\cdot)$ exists, and if $L^*(\mathcal{V}^*) \subset W$ is dense, $\mathbf{u} = L^*\mathbf{z}^* \in \mathrm{L}_2(Q;\mathbb{R}^m)$ is the unique weak solution of*

$$(\mathbf{u}, L^*\mathbf{z})_Q = \langle \ell, \mathbf{z}\rangle\,, \qquad \mathbf{z} \in V^*\,. \tag{30}$$

*Proof.* ad a) The functional $J(\cdot) > 0$ is bounded from below, and any minimizing sequence $\{\mathbf{u}_n\}_{n\in\mathbb{N}} \subset V$ with

$$\lim_{n\to\infty} J(\mathbf{u}_n) = \inf_{\mathbf{v}\in V} J(\mathbf{v}) := J_{\inf}$$

satisfies

$$\begin{aligned}
\frac{1}{4}\|L\mathbf{u}_n - L\mathbf{u}_k\|_{W^*}^2 &= \frac{1}{2}\|L\mathbf{u}_n - \mathbf{f}\|_{W^*}^2 + \frac{1}{2}\|L\mathbf{u}_k - \mathbf{f}\|_{W^*}^2 - \Big\|L\frac{1}{2}(\mathbf{u}_n + \mathbf{u}_k) - \mathbf{f}\Big\|_{W^*}^2\\
&= J(\mathbf{u}_n) + J(\mathbf{u}_k) - 2J\Big(\frac{1}{2}(\mathbf{u}_n + \mathbf{u}_k)\Big)\\
&\le J(\mathbf{u}_n) + J(\mathbf{u}_k) - 2J_{\inf} \longrightarrow 0 \quad \text{for } n, k \longrightarrow \infty\,.
\end{aligned}$$

Condition (27) implies the norm equivalence

$$\|L\mathbf{v}\|_{W^*} \le \|\mathbf{v}\|_V = \sqrt{\|\mathbf{v}\|_W^2 + \|L\mathbf{v}\|_{W^*}^2} \le \sqrt{1 + C_L^2}\,\|L\mathbf{v}\|_{W^*}\,, \qquad \mathbf{v} \in V\,, \tag{31}$$

so that the minimizing sequence is a Cauchy sequence converging to $\mathbf{u} \in V$. Since $J(\cdot)$ is strictly convex, the minimizer is unique. Moreover, since $J(\cdot)$ is differentiable, $\mathbf{u}$ is a critical point, i.e.,

$$0 = \partial J(\mathbf{u})[\mathbf{v}] = (L\mathbf{u} - \mathbf{f}, L\mathbf{v})_{W^*} = (L\mathbf{u} - \mathbf{f}, M^{-1}L\mathbf{v})_Q\,, \qquad \mathbf{v} \in V\,.$$

If $L$ is surjective, this implies (28) by inserting $\mathbf{w} = M^{-1}L\mathbf{v} \in M^{-1}L(V) = W$.
ad b) By assumption (29), $J^*$ and $\ell$ are continuous in $\mathcal{V}^*$ with respect to the norm in $V^*$, so they extend to $V^*$, and we observe that $J^*$ is bounded from below by

$$J^*(\mathbf{z}) = \frac{1}{2}\|L^*\mathbf{z}\|_{W^*}^2 - \langle \ell, \mathbf{z}\rangle \ge \frac{1}{2(1 + C_{L^*}^2)}\|\mathbf{z}\|_{V^*}^2 - C_\ell\,\|\mathbf{z}\|_{V^*} \ge -\frac{1}{2}C_\ell^2\big(1 + C_{L^*}^2\big)\,.$$

By the same arguments as above a unique minimizer $\mathbf{z}^* \in V^*$ exists characterized by

$$0 = \partial J^*(\mathbf{z}^*)[\mathbf{z}] = (L^*\mathbf{z}^*, L^*\mathbf{z})_{W^*} - \langle \ell, \mathbf{z}\rangle\,, \qquad \mathbf{z} \in V^*\,.$$

Inserting $\mathbf{u} = L^*\mathbf{z}^*$ implies (30). Now assume that $\tilde{\mathbf{u}}$ also solves (30); then, $(\mathbf{u} - \tilde{\mathbf{u}}, L^*\mathbf{z})_Q = 0$ for all $\mathbf{z} \in \mathcal{V}^*$. Since $L^*(\mathcal{V}^*)$ is dense in $W$, this implies $\mathbf{u} = \tilde{\mathbf{u}}$, so that the weak solution is unique. □

**Remark 12.** Strong solutions with inhomogeneous initial and boundary data exist, if the initial function $\mathbf{u}_0$ in $\Omega$ can be extended to a function $\mathbf{u}_0 \in \mathrm{H}(L, Q)$ satisfying the boundary conditions.

### 2.5. Mapping properties of the space-time operator

**Lemma 13.** *$\|\mathbf{v}\|_W \le C_L\|L\mathbf{v}\|_{W^*}$ for $\mathbf{v}\in V$ holds with $C_L = 2T$.*

*Proof.* For $\mathbf{v}\in\mathcal{V}$ we have $\mathbf{v}(0)=\mathbf{0}$, and using (23) we obtain

$$\begin{aligned}
\|\mathbf{v}\|_W^2 &= \int_0^T \big(M\mathbf{v}(t),\mathbf{v}(t)\big)_\Omega\,\mathrm{d}t = \int_0^T \Big(\big(M\mathbf{v}(t),\mathbf{v}(t)\big)_\Omega - \big(M\mathbf{v}(0),\mathbf{v}(0)\big)_\Omega\Big)\,\mathrm{d}t\\
&= \int_0^T\int_0^t \partial_s\big(M\mathbf{v}(s),\mathbf{v}(s)\big)_\Omega\,\mathrm{d}s\,\mathrm{d}t = 2\int_0^T\int_0^t \big(M\partial_s\mathbf{v}(s),\mathbf{v}(s)\big)_\Omega\,\mathrm{d}s\,\mathrm{d}t\\
&= 2\int_0^T\int_0^t \Big(\big(M\partial_s\mathbf{v}(s),\mathbf{v}(s)\big)_\Omega + \big(A\mathbf{v}(s),\mathbf{v}(s)\big)_\Omega\Big)\,\mathrm{d}s\,\mathrm{d}t\\
&= 2\int_0^T\int_0^t \big(L\mathbf{v}(s),\mathbf{v}(s)\big)_\Omega\,\mathrm{d}s\,\mathrm{d}t = 2\int_0^T (T-t)\big(L\mathbf{v}(t),\mathbf{v}(t)\big)_\Omega\,\mathrm{d}t\\
&\le 2T\,\|L\mathbf{v}\|_{W^*}\|\mathbf{v}\|_W\,.
\end{aligned}$$

Since $\mathcal{V}$ is dense in $V$, this extends to $V$. □

As a consequence of Lemma 13, the operator $L\colon V\to \mathrm{L}_2(Q;\mathbb{R}^m)$ is injective and continuous, i.e., $L\in\mathcal{L}(V,W)$.

**Corollary 14.** *$L(V)\subset \mathrm{L}_2(Q;\mathbb{R}^m)$ is closed.*

*Proof.* For any sequence $(\mathbf{w}_n)_{n\in\mathbb{N}}\subset V$ with $\lim\limits_{n\to\infty} L\mathbf{w}_n = \mathbf{f}\in W$ we have

$$\|\mathbf{w}_n-\mathbf{w}_k\|_W + \|L\mathbf{w}_n - L\mathbf{w}_k\|_{W^*} \le (C_L+1)\,\|L\mathbf{w}_n - L\mathbf{w}_k\|_{W^*} \longrightarrow 0\,,\quad n,k\to\infty\,,$$

so that $(\mathbf{w}_n)_n$ is a Cauchy sequence in $V$; since $V\subset \mathrm{H}(L,Q)$ is closed, the limit $\mathbf{w}=\lim \mathbf{w}_n\in V$ with $L\mathbf{w}=\mathbf{f}$ exists. □

Let the domain $\mathcal{D}(A) = Z\subset \mathrm{H}(A,\Omega)$ of the operator $A$ be the closure of $\mathcal{Z}$ defined in (22a). Then, (23) gives $\big((M+\tau A)\mathbf{z},\mathbf{z}\big)_\Omega = \big(M\mathbf{z},\mathbf{z}\big)_\Omega>0$ for all $\mathbf{z}\neq\mathbf{0}$ and $\tau\in\mathbb{R}$, i.e., $M+\tau A$ is injective on $Z$. Moreover, we require that $M+\tau A$ is surjective on $Z$, which is achieved in our applications in Sect. 2.7 by a suitable balanced selection of $\Gamma_k\subset\partial\Omega$.

**Lemma 15.** *Assume that $M+\tau A\colon Z\to \mathrm{L}_2(\Omega;\mathbb{R}^m)$ is surjective for all $\tau>0$.*
*Then, $L(V)\subset \mathrm{L}_2(Q;\mathbb{R}^m)$ is dense.*

*Proof.* For $\mathbf{f}\in \mathrm{L}_2(Q;\mathbb{R}^m)$, $N\in\mathbb{N}$ and $t_{N,n} = n\frac{T}{N}$ let $\mathbf{f}_N\in \mathrm{L}_2(Q;\mathbb{R}^m)$ be piecewise constant in time with $\mathbf{f}_{N,n} = \mathbf{f}_N|_{(t_{N,n-1},t_{N,n})}$ so that $\lim\limits_{N\to\infty}\|\mathbf{f}_N-\mathbf{f}\|_Q = 0$. Since the operator $M+\frac{T}{N}A\colon Z\longrightarrow \mathrm{L}_2(\Omega;\mathbb{R}^m)$ is surjective, starting with $\mathbf{u}_{N,0}=\mathbf{0}$ we find $\mathbf{u}_{N,n}\in Z$ with

$$\Big(M+\frac{T}{N}A\Big)\mathbf{u}_{N,n} = \mathbf{u}_{N,n-1} + \frac{T}{N}\mathbf{f}_{N,n}\,,\qquad n=1,\dots,N\,.$$

Let $\mathbf{u}_N\in \mathrm{H}^1(0,T;Z)\subset V$ be the piecewise linear interpolation: for $n=1,\dots,N$ set

$$\mathbf{u}_N(t) = \frac{t_{N,n}-t}{t_{N,n}-t_{N,n-1}}\mathbf{u}_{N,n-1} + \frac{t-t_{N,n-1}}{t_{N,n}-t_{N,n-1}}\mathbf{u}_{N,n}\,,\qquad t\in(t_{N,n-1},t_{N,n})\,.$$

Then, we observe by construction $L\mathbf{u}_N = \mathbf{f}_N$ and thus $\lim\limits_{N\to\infty}\|L\mathbf{u}_N-\mathbf{f}\|_Q = 0$. □

**Remark 16.** Together with Cor. 14 we observe $L(V) = \mathrm{L}_2(Q;\mathbb{R}^m)$, i.e., the operator $L\colon V\longrightarrow \mathrm{L}_2(Q;\mathbb{R}^m)$ is surjective.
A corresponding result can be achieved for $L^*(V^*)$ as the same arguments as in Lem. 13, 15, and Cor. 14 hold for $L^*$ and $V^*$. We obtain

$$\|\mathbf{z}\|_W \le C_L\|L^*\mathbf{z}\|_{W^*}\,,\qquad \mathbf{z}\in V^*$$

i.e., $C_L = C_{L^*}$. The operator $M-\tau A\colon Z\to \mathrm{L}_2(\Omega;\mathbb{R}^m)$ is surjective for all $\tau>0$, and $L^*(V^*)\subset \mathrm{L}_2(Q;\mathbb{R}^m)$ is dense which implies $L^*(V^*) = \mathrm{L}_2(\Omega;\mathbb{R}^m)$.

**Remark 17.** Since $L(\mathcal{V})$ and $L^*(\mathcal{V}^*)$ are dense in $W$, we have

$$V = \left\{ \mathbf{v} \in \mathrm{H}(L,Q) : (L\mathbf{v},\mathbf{z})_Q = (\mathbf{v}, L^*\mathbf{z})_Q \text{ for } \mathbf{z} \in \mathcal{V}^* \right\},$$
$$V^* = \left\{ \mathbf{z} \in \mathrm{H}(L^*,Q) : (L^*\mathbf{z},\mathbf{v})_Q = (\mathbf{z}, L\mathbf{v})_Q \text{ for } \mathbf{v} \in \mathcal{V} \right\},$$

i.e., $V^*$ is the Hilbert adjoint space of $V$, and $V$ is the Hilbert adjoint space of $V^*$.

**Lemma 18.** *For* $\mathbf{z} \in V^*$ *holds*

$$\|\mathbf{z}(0)\|_Y^2 \le \|\mathbf{z}\|_{V^*}^2 \,.$$

*Proof.* We obtain, using $\mathbf{z}(T) = \mathbf{0}$,

$$\begin{aligned} \|\mathbf{z}(0)\|_Y^2 &= \|\mathbf{z}(0)\|_Y^2 - \|\mathbf{z}(T)\|_Y^2 = -\int_0^T \partial_t \|\mathbf{z}(t)\|_Y^2 \,\mathrm{d}t = -2(M\partial_t\mathbf{z},\mathbf{z})_Q \\ &= -2(M\partial_t\mathbf{z},\mathbf{z})_Q - 2(A\mathbf{z},\mathbf{z})_Q = 2(L^*\mathbf{z},\mathbf{z})_Q \le \|\mathbf{z}\|_{V^*}^2 \,. \end{aligned}$$

□

### 2.6. **Inf-sup stability**

From the previous section we directly obtain the following results.

**Theorem 19.** *The bilinear form $b\colon V \times W \to \mathbb{R}$, $b(\mathbf{v},\mathbf{w}) = (L\mathbf{v},\mathbf{w})_Q$, is inf-sup stable satisfying*

$$\inf_{\mathbf{v}\in V\setminus\{\mathbf{0}\}} \sup_{\mathbf{w}\in W\setminus\{\mathbf{0}\}} \frac{b(\mathbf{v},\mathbf{w})}{\|\mathbf{v}\|_V\|\mathbf{w}\|_W} = \inf_{\mathbf{w}\in W\setminus\{\mathbf{0}\}} \sup_{\mathbf{v}\in V\setminus\{\mathbf{0}\}} \frac{b(\mathbf{v},\mathbf{w})}{\|\mathbf{v}\|_V\|\mathbf{w}\|_W} \ge \beta := \frac{1}{\sqrt{1+C_L^2}}\,.$$

*Thus, for all $\mathbf{f}\in \mathrm{L}_2(Q,\mathbb{R}^m)$ a unique Petrov–Galerkin solution $\mathbf{u}\in V$ of*

$$b(\mathbf{u},\mathbf{w}) = \big(\mathbf{f},\mathbf{w}\big)_Q\,, \qquad \mathbf{w}\in W,$$

*exists, and the solution is bounded by $\|\mathbf{u}\|_V \le \beta^{-1}\|\mathbf{f}\|_{W^*}$.*

*Proof.* For $\mathbf{v}\in V\setminus\{\mathbf{0}\}$ we test with $\mathbf{w} = M^{-1}L\mathbf{v}$, so that with (31)

$$\sup_{\mathbf{w}\in W\setminus\{\mathbf{0}\}} \frac{b(\mathbf{v},\mathbf{w})}{\|\mathbf{w}\|_W} \ge \frac{b(\mathbf{v},M^{-1}L\mathbf{v})}{\|M^{-1}L\mathbf{v}\|_W} = \|M^{-1}L\mathbf{v}\|_W \ge \frac{1}{\sqrt{1+C_L^2}}\|\mathbf{v}\|_V\,.$$

The existence and the a priori bound are now an easy consequence. □

**Corollary 20.** *Due to our previous results on the adjoint operator $L^*$ we find correspondingly that for all $\mathbf{d}\in \mathrm{L}_2(Q,\mathbb{R}^m)$ the dual problem $L^*\mathbf{z} = \mathbf{d}$ admits a unique solution $\mathbf{z}\in V^*$ which is bounded by $\|\mathbf{z}\|_{V^*} \le \beta^{-1}\|\mathbf{d}\|_{W^*}$.*

**Corollary 21.** *Additional regularity for the right-hand side* $\mathbf{f} \in \mathrm{H}^1(0,T;\mathrm{L}_2(\Omega;\mathbb{R}^m))$ *implies for the solution the regularity* $\mathbf{u} \in \mathrm{H}^1(0,T;\mathrm{L}_2(\Omega;\mathbb{R}^m))$ *and the estimate* $\|\partial_t \mathbf{u}\|_W \le C_L \|\partial_t \mathbf{f}\|_{W^*}$.

*Proof.* This simply follows from $L\mathbf{u} = \mathbf{f}$, which formally gives for the derivative in time $L\partial_t\mathbf{u} = \partial_t\mathbf{f}$. If $\partial_t\mathbf{f} \in W$, a solution $\mathbf{v} \in V$ solving $L\mathbf{v} = \partial_t\mathbf{f}$ exists, and since the solution is unique, $\mathbf{v} = \partial_t\mathbf{u}$. □

## 2.7. Applications to acoustics and visco-elasticity

**Acoustic waves.** In the setting of Example 6 we have $A(\mathbf{v},p) = -(\nabla p, \nabla\cdot\mathbf{v})$ and

$$\big(A(\mathbf{v},p),(\mathbf{w},q)\big)_\Omega + \big((\mathbf{v},p),A(\mathbf{w},q)\big)_\Omega = -(p,\mathbf{n}\cdot\mathbf{w})_{\partial\Omega} - (\mathbf{n}\cdot\mathbf{v},q)_{\partial\Omega}\,.$$

We now show that the assumption in Lem. 15 is satisfied. For all $(\mathbf{f},g) \in \mathrm{L}_2(\Omega;\mathbb{R}^{d+1})$ and $\tau > 0$ we define in the first step $p \in \mathrm{H}^1(\Omega)$ with $p = 0$ on $\Gamma_{\mathrm{S}}$ by solving the elliptic equation

$$\tau\big(\rho^{-1}\nabla p, \nabla\phi\big)_\Omega + \big(\kappa^{-1}p,\phi\big)_\Omega = \big(g,\phi\big)_\Omega - \big(\rho^{-1}\mathbf{f},\nabla\phi\big)_\Omega \tag{32}$$

for $\phi \in \mathrm{H}^1(\Omega)$ with $\phi = 0$ on $\Gamma_{\mathrm{S}}$. Then, we define $\mathbf{v} = \rho^{-1}(\tau\nabla p + \mathbf{f}) \in \mathrm{L}_2(\Omega;\mathbb{R}^d)$, and inserting (32), we observe

$$\big(\mathbf{v},\nabla\phi\big)_\Omega = \big(g,\phi\big)_\Omega - \big(\kappa^{-1}p,\phi\big)_\Omega\,, \qquad \phi \in \mathrm{C}^1_{\mathrm{c}}(\Omega)\,,$$

i.e., $\nabla\cdot\mathbf{v} = -g + \kappa^{-1}p \in \mathrm{L}_2(\Omega)$, and thus

$$0 = \big(\mathbf{v},\nabla\phi\big)_\Omega + \big(\nabla\cdot\mathbf{v},\phi\big)_\Omega = \langle \mathbf{n}\cdot\mathbf{v},\phi\rangle_{\partial\Omega}\,, \qquad \phi \in \mathrm{C}^1(\overline{\Omega})\,,\ \phi = 0 \text{ on } \Gamma_{\mathrm{S}}\,,$$

so that $\mathbf{n}\cdot\mathbf{v} = 0$ on $\partial\Omega\setminus\Gamma_{\mathrm{S}} = \Gamma_{\mathrm{V}}$. Together, $(\mathbf{v},p) \in Z$ and

$$(M+\tau A)(\mathbf{v},p) = (\mathbf{f},g)\,.$$

Moreover, the solution is unique, so that $M+\tau A$ is injective and surjective.

**Visco-elastic waves.** For the system (13) we set $\mathbf{y} = \big(\mathbf{v},\boldsymbol{\sigma}_0,\dots,\boldsymbol{\sigma}_r\big)^\top$ and

$$\underline{M} = \begin{pmatrix} \rho & 0 & \cdots & 0 \\ 0 & \mathbf{C}_0^{-1} & & \\ \vdots & & \ddots & \\ 0 & & & \mathbf{C}_r^{-1} \end{pmatrix}, \quad A = -\begin{pmatrix} 0 & \operatorname{div} & \cdots & \operatorname{div} \\ \boldsymbol{\varepsilon} & 0 & & \\ \vdots & & \ddots & \\ \boldsymbol{\varepsilon} & 0 & & 0 \end{pmatrix}, \quad \underline{D} = \underline{M}\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \tau_1^{-1} & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & & & \tau_r^{-1} \end{pmatrix}$$

with $m = 2+3(1+r)$ components for $d = 2$ and $m = 3+6(1+r)$ for $d = 3$, and where $\underline{D} \in \mathrm{L}_\infty(\Omega;\mathbb{R}^{m\times m}_{\mathrm{sym}})$ is a positive semi-definite matrix function. This defines the operator $D\mathbf{y}(x) = \underline{D}(x)\mathbf{y}(x)$, and we have $(D\mathbf{y},\mathbf{y})_\Omega \ge 0$ for all $\mathbf{y} \in \mathrm{L}_2(\Omega;\mathbb{R}^m)$.
The space-time setting is extended to the operator $L = M\partial_t + A + D$, and the formal adjoint operator is $L^* = -M\partial_t - A + D$. The assumption in Lem. 15 can be verified analogously to the acoustic case.

**Remark 22.** The extension to mixed boundary conditions on $\Gamma_{\mathrm{R}} \subset \partial\Omega$ requires $\mathrm{L}_2$ regularity of the traces on the boundary part $\Gamma_{\mathrm{R}}$. Then, extending the norm $\|\cdot\|_V$ by a corresponding boundary term again defines $V$ as closure of $\mathcal{V}$ with respect to this stronger norm, and the space-time operator $L$ is extended by a dissipative boundary operator $D$.

**Bibliographic comments.** Least squares for linear first-order systems for finite elements are considered in [Cai et al., 1994, Cai et al., 2001], where also the $LL^*$ technique is established which is used to prove Thm. 11 b). Here this is applied to the space-time setting, see [Dörfler et al., 2016, Dörfler et al., 2019, Ernesti and Wieners, 2019b, Ernesti and Wieners, 2019a]. The extension to mixed boundary conditions is considered in [Dörfler et al., 2020].
The inf-sup constant $\beta$ in Thm. 19 is not optimal for the continuous problem; for an improved estimate see [Ernesti and Wieners, 2019a, Lem. 1]. Here, it relies on the estimate for $C_L$ in Lem. 13 which is generalized in Thm. 26 for the approximation. The suitable choice of boundary conditions for general Friedrichs systems is discussed in [Di Pietro and Ern, 2011, Chap. 7.2].

# 3. Discontinuous Galerkin methods for linear hyperbolic systems

We develop a space-time method with a discontinuous Galerkin discretization in space for linear wave problems. For the ansatz space we use piecewise polynomials in every cell, where the traces on the cell interfaces can be different from the two sides. Therefore, we need to extend the first-order operator $A$ to discontinuous finite element spaces. Here, we introduce the discrete operator $A_h$ with upwind flux, where the evaluation of the upwind flux is based on solving Riemann problems, i.e., by construction of piecewise constant solutions in space and time. We start with simple examples for interface and transmission problems, and then consider the general case for waves in heterogeneous media.

## 3.1. Traveling wave solutions in homogeneous media

We consider linear hyperbolic first-order systems $L = M\partial_t + A$ introduced in Sect. 2.1, and we start with the case of homogeneous material parameters, so that the operator $M$ is represented by a symmetric positive definite matrix $\underline{M} \in \mathbb{R}^{m\times m}_{\mathrm{sym}}$ which is constant in $\overline{\Omega}$.
Let $(\lambda, \mathbf{w}) \in \mathbb{R}\times\mathbb{R}^m$ be an eigensystem of $\underline{A}_{\mathbf{n}}\mathbf{w} = \lambda\underline{M}\mathbf{w}$, and let $a \in \mathrm{C}^1(\mathbb{R})$ be an amplitude function describing the shape of the traveling wave. Then, we observe for $\mathbf{y}(t, \mathbf{x}) = a(\mathbf{n}\cdot\mathbf{x} - \lambda t)\,\mathbf{w}$

$$\begin{aligned}
\partial_t \mathbf{y}(t,\mathbf{x}) &= -\lambda a'(\mathbf{n}\cdot\mathbf{x} - \lambda t)\mathbf{w}\,,\\
\partial_{x_j} \mathbf{y}(t,\mathbf{x}) &= n_j a'(\mathbf{n}\cdot\mathbf{x} - \lambda t)\mathbf{w}\,,\\
L\mathbf{y}(t,\mathbf{x}) &= \underline{M}\partial_t\mathbf{y}(t,\mathbf{x}) + A\mathbf{y}(t,\mathbf{x})\\
&= a'(\mathbf{n}\cdot\mathbf{x} - \lambda t)\Big( -\lambda\underline{M} + \sum_{j=1}^{d} n_j\underline{A}_j\Big)\mathbf{w}\\
&= a'(\mathbf{n}\cdot\mathbf{x} - \lambda t)\Big(\underline{A}_{\mathbf{n}} - \lambda\underline{M}\Big)\mathbf{w} = \mathbf{0}\,,
\end{aligned}$$

so that $\mathbf{y}$ solves $L\mathbf{y} = \mathbf{0}$ for all $t \in \mathbb{R}$ in $\Omega = \mathbb{R}^d$.

**Example 23.** For acoustic waves with wave speed $c = \sqrt{\kappa/\rho}$ we have

$$\mathbf{y} = \begin{pmatrix}\mathbf{v}\\ p\end{pmatrix}, \quad \underline{M}\mathbf{y} = \begin{pmatrix}\rho\mathbf{v}\\ \kappa^{-1}p\end{pmatrix}, \quad \underline{A}_{\mathbf{n}}\mathbf{y} = -\begin{pmatrix}p\mathbf{n}\\ \mathbf{v}\cdot\mathbf{n}\end{pmatrix}, \quad \lambda \in \{0, \pm c\}\,, \quad \mathbf{w} = \begin{pmatrix}\mp c\mathbf{n}\\ \kappa\end{pmatrix}.$$

For elastic waves with wave speeds $c_{\mathrm{p}} = \sqrt{(2\mu + \lambda)/\rho}$ for compressional waves and $c_{\mathrm{s}} = \sqrt{\mu/\rho}$ for shear waves, we have

$$\begin{aligned}
&\mathbf{y} = \begin{pmatrix}\mathbf{v}\\ \boldsymbol{\sigma}\end{pmatrix}, \quad \underline{M}\mathbf{y} = \begin{pmatrix}\rho\mathbf{v}\\ \mathbf{C}^{-1}\boldsymbol{\sigma}\end{pmatrix}, \quad \underline{A}_{\mathbf{n}}\mathbf{y} = -\begin{pmatrix}\boldsymbol{\sigma}\mathbf{n}\\ \frac{1}{2}\big(\mathbf{n}\mathbf{v}^\top + \mathbf{v}\mathbf{n}^\top\big)\end{pmatrix},\\
&\lambda \in \{0, \pm c_{\mathrm{p}}, \pm c_{\mathrm{s}}\}\,, \quad \mathbf{w}_{\mathrm{p}} = \begin{pmatrix}\mp c_{\mathrm{p}}\mathbf{n}\\ 2\mu\mathbf{n}\mathbf{n}^\top + \lambda\mathbf{I}\end{pmatrix}, \quad \mathbf{w}_{\mathrm{s}} = \begin{pmatrix}\mp c_{\mathrm{s}}\boldsymbol{\tau}\\ \mu(\mathbf{n}\boldsymbol{\tau}^\top + \boldsymbol{\tau}\mathbf{n}^\top)\end{pmatrix},
\end{aligned}$$

where $\boldsymbol{\tau} \in \mathbb{R}^d$ is a tangential unit vector, i.e., $\boldsymbol{\tau}\cdot\mathbf{n} = 0$ and $|\boldsymbol{\tau}| = 1$.
For linear electro-magnetic waves with wave speed $c = 1/\sqrt{\varepsilon\mu}$ we have

$$\begin{aligned}
&\mathbf{y} = \begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}, \quad \underline{M}\mathbf{y} = \begin{pmatrix}\varepsilon\mathbf{E}\\ \mu\mathbf{H}\end{pmatrix}, \quad \underline{A}_{\mathbf{n}}\mathbf{y} = -\begin{pmatrix}\mathbf{n}\times\mathbf{H}\\ -\mathbf{n}\times\mathbf{E}\end{pmatrix},\\
&\lambda \in \{0, \pm c\}\,, \quad \mathbf{w}_1 = \begin{pmatrix}\sqrt{\varepsilon}\mathbf{n}\times\boldsymbol{\tau}\\ \pm\sqrt{\mu}\boldsymbol{\tau}\end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix}\pm\sqrt{\varepsilon}\boldsymbol{\tau}\\ \sqrt{\mu}\mathbf{n}\times\boldsymbol{\tau}\end{pmatrix}.
\end{aligned}$$

### 3.2. Reflection of traveling acoustic waves at boundaries

In the next step we consider solutions of the acoustic wave equation in the half space

$$\begin{pmatrix} \rho\partial_t \mathbf{v} - \nabla p \\ \kappa^{-1}\partial_t p - \nabla\cdot\mathbf{v} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix} \qquad \text{in } \Omega_{\mathrm{R}} = \big\{\mathbf{x}\in\mathbb{R}^d \colon \mathbf{n}\cdot\mathbf{x} > 0\big\}$$

with initial value

$$\begin{pmatrix} \mathbf{v}(0,\mathbf{x}) \\ p(0,\mathbf{x}) \end{pmatrix} = a(\mathbf{n}\cdot\mathbf{x}) \begin{pmatrix} c\mathbf{n} \\ \kappa \end{pmatrix}$$

depending on $a \in \mathrm{C}^1(\mathbb{R})$ with $a(\mathbf{n}\cdot\mathbf{x}) = 0$ for $\mathbf{n}\cdot\mathbf{x} < ct_0$ and $t_0 > 0$, i.e., $\operatorname{supp} a \subset [-\infty, ct_0]$.
The wave starts traveling from right to left, and at time $t = t_0$ it reaches the boundary. In case of a homogeneous Neumann boundary condition $\mathbf{v}\cdot\mathbf{n} = 0$ it is reflected, i.e.,

$$\begin{pmatrix} \mathbf{v}(t,\mathbf{x}) \\ p(t,\mathbf{x}) \end{pmatrix} = \begin{cases} a(ct+\mathbf{n}\cdot\mathbf{x}) \begin{pmatrix} c\mathbf{n} \\ \kappa \end{pmatrix} & 0 < c(t_0-t) < \mathbf{n}\cdot\mathbf{x}\,, \\ a(ct+\mathbf{n}\cdot\mathbf{x}) \begin{pmatrix} c\mathbf{n} \\ \kappa \end{pmatrix} + a(ct-\mathbf{n}\cdot\mathbf{x}) \begin{pmatrix} -c\mathbf{n} \\ \kappa \end{pmatrix} & 0 < \mathbf{n}\cdot\mathbf{x} < c(t-t_0)\,. \end{cases}$$

Otherwise, with homogeneous Dirichlet boundary conditions $p = 0$ the reflection also changes sign, i.e.,

$$\begin{pmatrix} \mathbf{v}(t,\mathbf{x}) \\ p(t,\mathbf{x}) \end{pmatrix} = \begin{cases} a(ct+\mathbf{n}\cdot\mathbf{x}) \begin{pmatrix} c\mathbf{n} \\ \kappa \end{pmatrix} & 0 < c(t_0-t) < \mathbf{n}\cdot\mathbf{x}\,, \\ a(ct+\mathbf{n}\cdot\mathbf{x}) \begin{pmatrix} c\mathbf{n} \\ \kappa \end{pmatrix} - a(ct-\mathbf{n}\cdot\mathbf{x}) \begin{pmatrix} -c\mathbf{n} \\ \kappa \end{pmatrix} & 0 < \mathbf{n}\cdot\mathbf{x} < c(t-t_0)\,. \end{cases}$$

For smooth amplitude functions this is a classical solution.

### 3.3. Transmission and reflection of traveling waves at interfaces

Now we consider solutions of the acoustic wave equation in $\mathbb{R}^d$ with an interface

$$\begin{pmatrix} \rho\partial_t \mathbf{v} - \nabla p \\ \kappa^{-1}\partial_t p - \nabla\cdot\mathbf{v} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix} \qquad \text{in } \Omega_{\mathrm{L}}\cup\Omega_{\mathrm{R}}\,, \qquad \begin{cases} \Omega_{\mathrm{L}} = \big\{\mathbf{x}\in\mathbb{R}^d \colon \mathbf{n}\cdot\mathbf{x} < 0\big\}, \\ \Omega_{\mathrm{R}} = \big\{\mathbf{x}\in\mathbb{R}^d \colon \mathbf{n}\cdot\mathbf{x} > 0\big\} \end{cases}$$

with constant coefficients $(\rho_{\mathrm{L}}, \kappa_{\mathrm{L}})$ in $\Omega_{\mathrm{L}}$ and $(\rho_{\mathrm{R}}, \kappa_{\mathrm{R}})$ in $\Omega_{\mathrm{R}}$ defining $\underline{M}_{\mathrm{L}}$ and $\underline{M}_{\mathrm{R}}$, starting in $\Omega_{\mathrm{R}}$ with

$$\begin{pmatrix} \mathbf{v}(0,\mathbf{x}) \\ p(0,\mathbf{x}) \end{pmatrix} = a(\mathbf{n}\cdot\mathbf{x}/c_{\mathrm{R}}) \begin{pmatrix} \mathbf{n} \\ Z_{\mathrm{R}} \end{pmatrix}\,, \qquad a(\mathbf{n}\cdot\mathbf{x}) = 0 \text{ for } \mathbf{n}\cdot\mathbf{x} < c_{\mathrm{R}} t_0\,,\ t_0 > 0\,,$$

where $Z_{\mathrm{L}} = \sqrt{\kappa_{\mathrm{L}}\rho_{\mathrm{L}}}$, $Z_{\mathrm{R}} = \sqrt{\kappa_{\mathrm{R}}\rho_{\mathrm{R}}}$ are the left and right impedances, and where $c_{\mathrm{L}} = \sqrt{\kappa_{\mathrm{L}}/\rho_{\mathrm{L}}}$, $c_{\mathrm{R}} = \sqrt{\kappa_{\mathrm{R}}/\rho_{\mathrm{R}}}$ are the left and right wave speeds. Note that we use a different scaling of the eigenvectors for the transmission problem.
We state continuity at the interface to determine a classical solution and obtain

$$\begin{pmatrix} \mathbf{v}(t,\mathbf{x}) \\ p(t,\mathbf{x}) \end{pmatrix} = \begin{cases} a(t+\mathbf{n}\cdot\mathbf{x}/c_{\mathrm{R}}) \begin{pmatrix} \mathbf{n} \\ Z_{\mathrm{R}} \end{pmatrix} & 0 < c_{\mathrm{R}}(t_0-t) < \mathbf{n}\cdot\mathbf{x}, \\ a(t+\mathbf{n}\cdot\mathbf{x}/c_{\mathrm{R}}) \begin{pmatrix} \mathbf{n} \\ Z_{\mathrm{R}} \end{pmatrix} & \\ \qquad + \beta_{\mathrm{R}}\, a(t-\mathbf{n}\cdot\mathbf{x}/c_{\mathrm{R}}) \begin{pmatrix} -\mathbf{n} \\ Z_{\mathrm{R}} \end{pmatrix} & 0 < \mathbf{n}\cdot\mathbf{x} < c_{\mathrm{R}}(t-t_0), \\ \beta_{\mathrm{L}}\, a(t+\mathbf{n}\cdot\mathbf{x}/c_{\mathrm{L}}) \begin{pmatrix} \mathbf{n} \\ Z_{\mathrm{L}} \end{pmatrix} & c_{\mathrm{L}}(t_0-t) < \mathbf{n}\cdot\mathbf{x} < 0 \end{cases}$$

with transmission and reflection coefficients

$$\beta_{\rm R} = \frac{2Z_{\rm R}}{Z_{\rm R}+Z_{\rm L}}\,, \qquad \beta_{\rm L} = \frac{Z_{\rm L}-Z_{\rm R}}{Z_{\rm R}+Z_{\rm L}}$$

derived from the interface condition $\underline{A}_{\mathbf{n}}[\mathbf{y}] = \mathbf{0}$; by the interface condition we obtain $L(\mathbf{v}, p) \in \mathrm{L}_{2,\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}^{d+1})$, so that $(\mathbf{v}, p)$ is a strong solution.
We observe that no wave is reflected if the impedance $Z_{\rm L} = Z_{\rm R}$ is continuous. This properties can be used to design absorbing boundary layers.

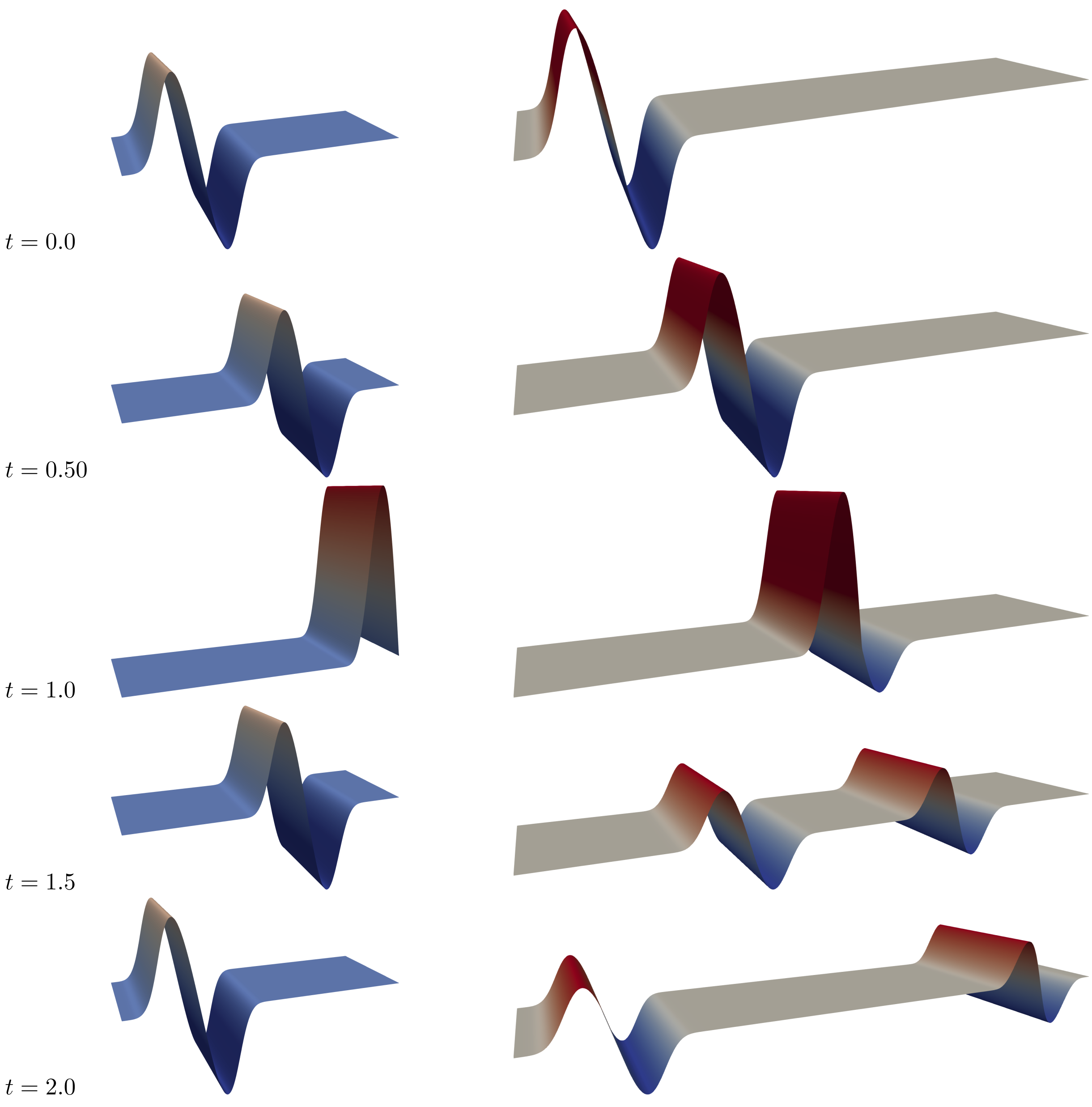



FIGURE 3. The evolution of the pressure distribution with reflection at a fixed boundary (left, cf. Sect. 3.2), and reflection and transmission at an interface (right, cf. Sect. 3.3) of traveling waves.

### 3.4. **The Riemann problem for acoustic waves**

Now we consider weak solutions in $\mathrm{L}_{2,\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^{d+1})$ of the acoustic wave equation

$$\begin{pmatrix}\rho\partial_t\mathbf{v}-\nabla p\\ \kappa^{-1}\partial_t p-\nabla\cdot\mathbf{v}\end{pmatrix}=\begin{pmatrix}\mathbf{0}\\0\end{pmatrix}\qquad\text{in }\Omega_\mathrm{L}\cup\Omega_\mathrm{R}\,,\qquad\begin{cases}\Omega_\mathrm{L}=\big\{\mathbf{x}\in\mathbb{R}^d\colon\mathbf{n}\cdot\mathbf{x}<0\big\}\\ \Omega_\mathrm{R}=\big\{\mathbf{x}\in\mathbb{R}^d\colon\mathbf{n}\cdot\mathbf{x}>0\big\}\end{cases}\,,$$

with constant coefficients $(\rho_\mathrm{L},\kappa_\mathrm{L})$ in $\Omega_\mathrm{L}$ and $(\rho_\mathrm{R},\kappa_\mathrm{R})$ in $\Omega_\mathrm{R}$, and with piecewise constant initial values

$$\begin{pmatrix}\mathbf{v}(0,\mathbf{x})\\p(0,\mathbf{x})\end{pmatrix}=\begin{pmatrix}\mathbf{v}_\mathrm{L}\\p_\mathrm{L}\end{pmatrix}\,,\quad\mathbf{x}\in\Omega_\mathrm{L}\,,\qquad\begin{pmatrix}\mathbf{v}(0,\mathbf{x})\\p(0,\mathbf{x})\end{pmatrix}=\begin{pmatrix}\mathbf{v}_\mathrm{R}\\p_\mathrm{R}\end{pmatrix}\,,\quad\mathbf{x}\in\Omega_\mathrm{R}\,,$$

called *Riemann problem.* The weak solution is of the form

$$\begin{pmatrix}\mathbf{v}(t,\mathbf{x})\\p(t,\mathbf{x})\end{pmatrix}=\begin{cases}\begin{pmatrix}\mathbf{v}_\mathrm{L}\\p_\mathrm{L}\end{pmatrix} & \mathbf{x}\cdot\mathbf{n}<-c_\mathrm{L}t\\ \begin{pmatrix}\mathbf{v}_\mathrm{L}\\p_\mathrm{L}\end{pmatrix}+\beta_\mathrm{L}\begin{pmatrix}\mathbf{n}\\Z_\mathrm{L}\end{pmatrix} & -c_\mathrm{L}t<\mathbf{x}\cdot\mathbf{n}<0\\ \begin{pmatrix}\mathbf{v}_\mathrm{R}\\p_\mathrm{R}\end{pmatrix}+\beta_\mathrm{R}\begin{pmatrix}\mathbf{n}\\-Z_\mathrm{R}\end{pmatrix} & 0<\mathbf{x}\cdot\mathbf{n}<c_\mathrm{R}t\\ \begin{pmatrix}\mathbf{v}_\mathrm{R}\\p_\mathrm{R}\end{pmatrix} & c_\mathrm{R}t<\mathbf{x}\cdot\mathbf{n}\end{cases}$$

depending on $\beta_\mathrm{L},\beta_\mathrm{R}\in\mathbb{R}$ determined by the flux condition

$$\underline{A}_\mathbf{n}\left(\begin{pmatrix}\mathbf{v}_\mathrm{L}\\p_\mathrm{L}\end{pmatrix}+\beta_\mathrm{L}\begin{pmatrix}\mathbf{n}\\Z_\mathrm{L}\end{pmatrix}\right)=\underline{A}_\mathbf{n}\left(\begin{pmatrix}\mathbf{v}_\mathrm{R}\\p_\mathrm{R}\end{pmatrix}+\beta_\mathrm{R}\begin{pmatrix}\mathbf{n}\\-Z_\mathrm{R}\end{pmatrix}\right)\,,\tag{33}$$

which yields $\beta_\mathrm{L}=\dfrac{[p]+Z_\mathrm{R}\mathbf{n}\cdot[\mathbf{v}]}{Z_\mathrm{L}+Z_\mathrm{R}}$, $\beta_\mathrm{R}=\dfrac{[p]-Z_\mathrm{L}\mathbf{n}\cdot[\mathbf{v}]}{Z_\mathrm{L}+Z_\mathrm{R}}$ depending on $[p]=p_\mathrm{R}-p_\mathrm{L}$, $[\mathbf{v}]=\mathbf{v}_\mathrm{R}-\mathbf{v}_\mathrm{L}$.
For discontinuous initial values the solution is discontinuous along the characteristic linear manifolds $\mathbf{x}\cdot\mathbf{n}+c_\mathrm{L}t=0$ and $\mathbf{x}\cdot\mathbf{n}-c_\mathrm{R}t=0$ in the space-time domain, so that we only obtain a weak solution.

### 3.5. **The Riemann problem for linear conservation laws**

We now construct a weak solution of the Riemann problem for general linear conservation laws, i.e., a piecewise constant weak solution of $L\mathbf{y}=\mathbf{0}$ in $\mathrm{L}_{2,\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^m)$ with discontinuous initial values

$$\mathbf{y}_0(\mathbf{x})=\begin{cases}\mathbf{y}_\mathrm{L} & \text{in }\Omega_\mathrm{L}=\big\{\mathbf{x}\in\mathbb{R}^d\colon\mathbf{n}\cdot\mathbf{x}<0\big\},\\ \mathbf{y}_\mathrm{R} & \text{in }\Omega_\mathrm{R}=\big\{\mathbf{x}\in\mathbb{R}^d\colon\mathbf{n}\cdot\mathbf{x}>0\big\},\end{cases}\quad\mathbf{y}_\mathrm{L},\mathbf{y}_\mathrm{R}\in\mathbb{R}^m\,,\quad\underline{M}_\mathrm{L}\,,\underline{M}_\mathrm{R}\in\mathbb{R}^{m\times m}_\mathrm{sym}\,.$$

Let $\big\{(\lambda_j^\mathrm{L},\mathbf{w}_j^\mathrm{L})\big\}_{j=1,\dots,m}$ and $\big\{(\lambda_j^\mathrm{R},\mathbf{w}_j^\mathrm{R})\big\}_{j=1,\dots,m}$ be eigenpairs, i.e.,

$$\underline{A}_\mathbf{n}\mathbf{w}_j^\mathrm{L}=\lambda_j^\mathrm{L}\underline{M}_\mathrm{L}\mathbf{w}_j^\mathrm{L}\,,\ \underline{A}_\mathbf{n}\mathbf{w}_j^\mathrm{R}=\lambda_j^\mathrm{R}\underline{M}_\mathrm{R}\mathbf{w}_j^\mathrm{R}\,,\quad\mathbf{w}_k^\mathrm{L}\cdot\underline{M}_\mathrm{L}\mathbf{w}_j^\mathrm{L}=\mathbf{w}_k^\mathrm{R}\cdot\underline{M}_\mathrm{R}\mathbf{w}_j^\mathrm{R}=0\text{ for }j\neq k\,.$$

A solution is constructed by a superposition of traveling waves

$$\mathbf{y}(t,\mathbf{x})=\begin{cases}\mathbf{y}_\mathrm{L}+\displaystyle\sum_{j\colon\mathbf{x}\cdot\mathbf{n}>\lambda_j^\mathrm{L}t}\beta_j^\mathrm{L}\mathbf{w}_j^\mathrm{L} & \mathbf{x}\in\Omega_\mathrm{L},\\ \mathbf{y}_\mathrm{R}+\displaystyle\sum_{j\colon\mathbf{x}\cdot\mathbf{n}<\lambda_j^\mathrm{R}t}\beta_j^\mathrm{R}\mathbf{w}_j^\mathrm{R} & \mathbf{x}\in\Omega_\mathrm{R},\end{cases}$$

and by solving the equation for $\beta_j^\mathrm{L}$, $\beta_j^\mathrm{R}$ (only depending on $[\mathbf{y}_0]=\mathbf{y}_\mathrm{R}-\mathbf{y}_\mathrm{L}$)

$$\underline{A}_\mathbf{n}\Big(\mathbf{y}_\mathrm{L}+\sum_{j\colon\lambda_j^\mathrm{L}<0}\beta_j^\mathrm{L}\mathbf{w}_j^\mathrm{L}\Big)=\underline{A}_\mathbf{n}\Big(\mathbf{y}_\mathrm{R}+\sum_{j\colon\lambda_j^\mathrm{R}>0}\beta_j^\mathrm{R}\mathbf{w}_j^\mathrm{R}\Big)\quad\text{on}\quad\partial\Omega_\mathrm{L}\cap\partial\Omega_\mathrm{R}\,.\tag{34}$$

Then, the flux $\underline{A}_\mathbf{n}\mathbf{y}$ is continuous for $t>0$, and the piecewise constant function $\mathbf{y}$ is the unique weak solution of $L\mathbf{y}=\mathbf{0}$ with initial value $\mathbf{y}(0)=\mathbf{y}_0$.

In summary, the solution of the Riemann problem defines the *upwind flux*

$$A_{\mathbf{n}}^{\mathrm{upw}}\mathbf{y}_0 = \underline{A}_{\mathbf{n}}\Big(\mathbf{y}_{\mathrm{L}} + \sum_{j\colon \lambda_j^{\mathrm{L}}<0} \beta_j^{\mathrm{L}}\mathbf{w}_j^{\mathrm{L}}\Big)\,. \tag{35}$$

On the boundary, depending on the boundary conditions, a system corresponding to (34) is solved defining an operator $\underline{A}_{\mathbf{n}}^{\mathrm{bnd}}$ with

$$A_{\mathbf{n}}^{\mathrm{upw}}\mathbf{y}_0 = \underline{A}_{\mathbf{n}}\mathbf{y}_{\mathrm{L}} + \underline{A}_{\mathbf{n}}^{\mathrm{bnd}}\mathbf{g} \tag{36}$$

depending on the boundary data $\mathbf{g}$. This is specified in the following sections for our examples.

### 3.6. **The DG discretization with full upwind**

Let $\mathcal{K}_h$ be a set of open convex cells $K\in\mathcal{K}_h$ with $K\subset\Omega\subset\mathbb{R}^d$ such that $\Omega_h = \bigcup\limits_{K\in\mathcal{K}_h} K$ is a decomposition of $\Omega$ with skeleton $\partial\Omega_h = \overline{\Omega}\setminus\Omega_h = \bigcup\limits_{K\in\mathcal{K}_h}\partial K$.

Let $\mathcal{F}_K$ be the set of faces $F\subset\partial K$, such that $\overline{F} = \partial K\cap\partial\Omega$ for boundary faces, and such that for inner faces $F\subset\partial K\cap\Omega$ the neighboring cell $K_F\in\mathcal{K}_h$ exists with $\overline{F} = \partial K\cap\partial K_F$. Let $\mathcal{F}_h = \bigcup_K \mathcal{F}_K$ be the set of all faces. For the boundary conditions on $\Gamma_k\subset\partial\Omega$ we assume compatibility of the decomposition so that $\overline{\Gamma}_k = \bigcup_{F\in\mathcal{F}_h\cap\Gamma_k}\overline{F}$.

Let $Y_h\subset\mathbb{P}(\Omega_h;\mathbb{R}^m) = \prod_{K\in\mathcal{K}_h}\mathbb{P}(K;\mathbb{R}^m)$ be a discontinuous piecewise polynomial finite element space, where $\mathbb{P}(K)$ denotes the space of polynomials of any degree in $K$.

For $\mathbf{y}_h\in Y_h$, let $\mathbf{y}_{h,K}\in\mathbb{P}(\overline{K};\mathbb{R}^m)$ be the continuous extension of $\mathbf{y}_h|_K$ to $\overline{K}$. On inner faces $F\in\mathcal{F}_h\cap\Omega$, we define by $[\mathbf{y}_h]_{K,F} = \mathbf{y}_{h,K_F} - \mathbf{y}_{h,K}$ the *jump across* $F$.

**Lemma 24.** *We have for* $\mathbf{y}_h \in Y_h$*:*

$$\mathbf{y}_h \in \mathrm{H}(A,\mathbb{R}^m) \quad \Longleftrightarrow \quad \underline{A}_{\mathbf{n}_K}[\mathbf{y}_h]_{K,F} = \mathbf{0} \quad \textit{for all } F \in \mathcal{F}_K \cap \Omega\,,\ K \in \mathcal{K}_h\,.$$

*Proof.* We define $\mathbf{f}_{h,K} = A\mathbf{y}_{h,K}$ in $K$, and since $|\Omega \setminus \Omega_h|_d = 0$, this defines a function $\mathbf{f}_h \in \mathrm{L}_2(\Omega;\mathbb{R}^m)$ by $\mathbf{f}_h|_K = \mathbf{f}_{h,K}$. Now we observe for test functions $\mathbf{z} \in \mathrm{C}^1_\mathrm{c}(\Omega,\mathbb{R}^m)$

$$\big(\mathbf{f}_h,\mathbf{z}\big)_\Omega + \big(\mathbf{y}_h, A\mathbf{z}\big)_\Omega = \sum_{K\in\mathcal{K}_h}\Big(\big(\mathbf{f}_{h,K} - A\mathbf{y}_{h,K},\mathbf{z}\big)_K + \big(\underline{A}_{\mathbf{n}_K}\mathbf{y}_{h,K},\mathbf{z}\big)_{\partial K}\Big) = -\frac{1}{2}\sum_{K\in\mathcal{K}_h}\sum_{F\in\mathcal{F}_K\cap\Omega}\big(\underline{A}_{\mathbf{n}_K}[\mathbf{y}_h]_{K,F},\mathbf{z}\big)_F$$

using $\underline{A}_{\mathbf{n}_{K_F}} = -\underline{A}_{\mathbf{n}_K}$. Thus, $\mathbf{y}_h \in \mathrm{H}(A,\mathbb{R}^m)$ and $A\mathbf{y}_h = \mathbf{f}_h \in \mathrm{L}_2(\Omega;\mathbb{R}^m)$ if and only if $\underline{A}_{\mathbf{n}_K}[\mathbf{y}_h]_{K,F}$ vanishes on all inner faces. □

For $\mathbf{y}_h, \mathbf{z}_h \in Y_h$ we observe

$$\big(A\mathbf{y}_h,\mathbf{z}_h\big)_\Omega = \sum_{K\in\mathcal{K}_h}\big(\operatorname{div}\underline{A}\mathbf{y}_{h,K},\mathbf{z}_{h,K}\big)_K = \sum_{K\in\mathcal{K}_h}\Big(\big(\underline{A}_{\mathbf{n}_K}\mathbf{y}_{h,K},\mathbf{z}_{h,K}\big)_{\partial K} - \big(\mathbf{y}_{h,K}, A\mathbf{z}_{h,K}\big)_K\Big)\,.$$

Inserting the upwind flux (35) defines the DG approximation $A_h$, where $\underline{A}_{\mathbf{n}_K}$ is replaced by $\underline{A}^{\mathrm{upw}}_{\mathbf{n}_K}\mathbf{y}_h$, i.e.,

$$\begin{aligned}\big(A_h\mathbf{y}_h,\mathbf{z}_h\big)_\Omega &= \sum_{K\in\mathcal{K}_h}\Big(\big(\underline{A}^{\mathrm{upw}}_{\mathbf{n}_K}\mathbf{y}_h,\mathbf{z}_{h,K}\big)_{\partial K} - \big(\mathbf{y}_{h,K}, A\mathbf{z}_{h,K}\big)_K\Big) \\ &= \sum_{K\in\mathcal{K}_h}\bigg(\big(A\mathbf{y}_{h,K},\mathbf{z}_{h,K}\big)_K + \sum_{F\in\mathcal{F}_K}\big(\underline{A}^{\mathrm{upw}}_{\mathbf{n}_K}\mathbf{y}_h - \underline{A}_{\mathbf{n}_K}\mathbf{y}_{h,K},\mathbf{z}_{h,K}\big)_F\bigg)\,.\end{aligned} \tag{37}$$

For inhomogeneous boundary conditions, using (36), the corresponding right-hand side is defined by

$$\langle \ell_h, \mathbf{z}_h\rangle = \big(\mathbf{f},\mathbf{z}_h\big)_\Omega - \sum_{F\in\mathcal{F}_h\cap\partial\Omega}\big(\underline{A}^{\mathrm{bnd}}_{\mathbf{n}}\mathbf{g},\mathbf{z}_h\big)_F\,. \tag{38}$$

As we see in our examples, the boundary term is consistent with

$$\big(\underline{A}^{\mathrm{bnd}}_{\mathbf{n}}\mathbf{g}_h,\mathbf{z}\big)_{(0,T)\times\partial\Omega} = \sum_{k=1}^m\big(g_k, z_k\big)_{(0,T)\times\Gamma_k} \tag{39}$$

for all test functions $\mathbf{z} \in \mathcal{D}(A)$ with homogeneous boundary conditions $z_k = 0$ on $\partial\Omega\setminus\Gamma_k$, $k = 1,\dots,m$.

**Consistent extension of the discrete DG operator.** For sufficiently smooth functions $\mathbf{y} \in \mathrm{H}^1(\Omega_h;\mathbb{R}^m)$ traces on the skeleton $\partial\Omega_h$exist in $\mathrm{L}_2$, so that the discrete operator $A_h$ extends to $A_h \in \mathcal{L}(\mathrm{H}^1(\Omega_h;\mathbb{R}^m), Y_h)$ by

$$\big(A_h\mathbf{y},\mathbf{z}_h\big)_\Omega = \sum_{K\in\mathcal{K}_h}\Big(\big(A\mathbf{y},\mathbf{z}_{h,K}\big)_K + \sum_{F\in\mathcal{F}_K}\big(\underline{A}^{\mathrm{upw}}_{\mathbf{n}_K}\mathbf{y} - \underline{A}_{\mathbf{n}_K}\mathbf{y},\mathbf{z}_{h,K}\big)_F\Big) \tag{40}$$

for $\mathbf{y} \in \mathrm{H}^1(\Omega_h;\mathbb{R}^m)$ and $\mathbf{z}_h \in Y_h$. Since in the conforming case by construction

$$\underline{A}^{\mathrm{upw}}_{\mathbf{n}_K}\mathbf{y} = \underline{A}_{\mathbf{n}_K}\mathbf{y}_K\,,\qquad K\in\mathcal{K}_h\,,\ \mathbf{y}\in\mathrm{H}^1(\Omega;\mathbb{R}^m)\,,$$

we obtain consistency for sufficiently smooth functions, i.e.,

$$\big(A_h\mathbf{y},\mathbf{z}_h\big)_\Omega = \big(A\mathbf{y},\mathbf{z}_h\big)_\Omega\,,\qquad \mathbf{y}\in\mathrm{H}^1_0(\Omega;\mathbb{R}^m)\,,\ \mathbf{z}_h\in Y_h\,. \tag{41}$$

In case of homogeneous boundary conditions, this extends to $\mathbf{y} \in Y_h \cap \mathcal{D}(A)$; this will be proved for acoustics in Lem. 25.

### 3.7. The full upwind discretization for the wave equation

We evaluate (35) using the eigensystems in Example 23. Therefore, we assume that the material parameters are constant in every cell $K$ and the possible material interfaces are aligned with the mesh.
For acoustics, we obtain on inner faces $F \in \mathcal{F}_h \cap \Omega$ from (33)

$$\begin{aligned}
&\underline{A}_{\mathbf{n}_K}\left(\begin{pmatrix}\mathbf{v}_{h,K}\\ p_{h,K}\end{pmatrix} + \beta_K\begin{pmatrix}\mathbf{n}_K\\ Z_K\end{pmatrix}\right) = \underline{A}_{\mathbf{n}_K}\left(\begin{pmatrix}\mathbf{v}_{h,K_F}\\ p_{h,K_F}\end{pmatrix} + \beta_{K_F}\begin{pmatrix}\mathbf{n}_K\\ -Z_{K_F}\end{pmatrix}\right)\\
\Longrightarrow\quad & 0 = \begin{pmatrix}\mathbf{n}_K\\ Z_{K_F}\end{pmatrix}\cdot\underline{A}_{\mathbf{n}_K}\left(\begin{pmatrix}[\mathbf{v}_h]_{K,F}\\ [p_h]_{K,F}\end{pmatrix} - \beta_K\begin{pmatrix}\mathbf{n}_K\\ Z_K\end{pmatrix}\right)\\
\Longrightarrow\quad & \underline{A}^{\mathrm{upw}}_{\mathbf{n}_K}\begin{pmatrix}\mathbf{v}_h\\ p_h\end{pmatrix} = \underline{A}_{\mathbf{n}_K}\begin{pmatrix}\mathbf{v}_{h,K}\\ p_{h,K}\end{pmatrix} - \frac{[p_h]_{K,F} + Z_{K_F}\mathbf{n}_K\cdot[\mathbf{v}_h]_{K,F}}{Z_K + Z_{K_F}}\begin{pmatrix}Z_K\mathbf{n}_K\\ 1\end{pmatrix}
\end{aligned} \tag{42}$$

by solving the equation for $\beta_K$. This extends to the boundary by defining the jump terms depending on the boundary conditions. On boundary faces $F \in \mathcal{F}_h \cap \partial\Omega$, we obtain from

$$\underline{A}_{\mathbf{n}_K}\left(\begin{pmatrix}\mathbf{v}_K\\ p_K\end{pmatrix} + \beta_K\begin{pmatrix}\mathbf{n}_K\\ Z_K\end{pmatrix}\right) = \begin{pmatrix}p_K\mathbf{n}_K\\ \mathbf{n}_K\cdot\mathbf{v}_K\end{pmatrix} - \beta_K\begin{pmatrix}Z_K\mathbf{n}_K\\ 1\end{pmatrix} \tag{43}$$

in case of Dirichlet boundary conditions $\beta_K = \frac{1}{Z_K}$, which corresponds to the numerical fluxes $[p_h]_{K,F} = -2p_h$ and $\mathbf{n}_K \cdot [\mathbf{v}_h]_{K,F} = 0$. This applies to the static boundary $\Gamma_{\mathrm{S}}$ for the pressure (14f).
In case of Neumann boundary conditions we obtain $\beta_K = 1$ corresponding to $\mathbf{n}_K \cdot [\mathbf{v}_h]_{K,F} = -2\mathbf{n}_K \cdot \mathbf{v}_h$ and $[p_h]_{K,F} = 0$, which applies to the dynamic boundary $\Gamma_{\mathrm{V}}$ for the velocity (14e). In both cases we extend the impedance on boundary faces $F$ by $Z_{K_F} = Z_K$.

**The DG operator for acoustics and visco-acoustics.** The operator $A_h \in \mathcal{L}\big(Y_h, Y_h\big)$, $A_h = \sum_{K\in\mathcal{K}_h} A_{h,K}$ for acoustics (with $r = 0$) and visco-acoustics ($r \geq 1$) with full upwind (42) on inner faces and (43) on the boundary is explicitly given by

$$\begin{aligned}
\big(A_{h,K}\mathbf{y}_h,\mathbf{z}_h\big)_K =\ & -\big(\nabla\cdot\mathbf{v}_{h,K},q_{h,K}\big)_K - \big(\nabla p_{h,K},\mathbf{w}_{h,K}\big)_K\\
&- \sum_{F\in\mathcal{F}_K}\frac{1}{Z_K+Z_{K_F}}\big([p_h]_{K,F} + Z_{K_F}\mathbf{n}_K\cdot[\mathbf{v}_h]_{K,F}, q_{h,K} + Z_K\mathbf{n}_K\cdot\mathbf{w}_{h,K}\big)_F\\
=\ & -\big(\nabla\cdot\mathbf{v}_{h,K},q_{h,K}\big)_K - \big(\nabla p_{h,K},\mathbf{w}_{h,K}\big)_K\\
&- \sum_{F\in\mathcal{F}_K\cap\Omega}\frac{1}{Z_K+Z_{K_F}}\big([p_h]_{K,F} + Z_{K_F}\mathbf{n}_K\cdot[\mathbf{v}_h]_{K,F}, q_{h,K} + Z_K\mathbf{n}_K\cdot\mathbf{w}_{h,K}\big)_F\\
&+ \sum_{F\in\mathcal{F}_K\cap\Gamma_{\mathrm{S}}}\frac{1}{Z_K}\big(p_{h,K}, q_{h,K} + Z_K\mathbf{n}_K\cdot\mathbf{w}_{h,K}\big)_F\\
&+ \sum_{F\in\mathcal{F}_K\cap\Gamma_{\mathrm{V}}}\big(\mathbf{n}_K\cdot\mathbf{v}_{h,K}, q_{h,K} + Z_K\mathbf{n}_K\cdot\mathbf{w}_{h,K}\big)_F
\end{aligned} \tag{44}$$

for $\mathbf{y}_h = (\mathbf{v}_h, p_{0,h},\dots,p_{r,h})$, $\mathbf{z}_h = (\mathbf{w}_h, q_{0,h},\dots,q_{r,h}) \in Y_h$ with $p_h = p_{0,h} + \cdots + p_{r,h}$, $q_h = q_{0,h} + \cdots + q_{r,h}$.

For inhomogeneous boundary conditions we obtain the right-hand side (38) by $\ell_h = \sum_{K\in\mathcal{K}_h} \ell_{h,K}$ with

$$\langle \ell_{h,K}, \mathbf{z}_{h,K}\rangle = \big(\mathbf{f}_h, \mathbf{z}_{h,K}\big)_K + \sum_{F\in\mathcal{F}_K\cap\Gamma_\mathrm{S}} \big(p_\mathrm{S}, Z_K^{-1} q_{h,K} + \mathbf{n}_K\cdot\mathbf{w}_{h,K}\big)_F + \sum_{F\in\mathcal{F}_K\cap\Gamma_\mathrm{V}} \big(g_\mathrm{V}, q_{h,K} + Z_K \mathbf{n}_K\cdot\mathbf{w}_{h,K}\big)_F\,. \tag{45}$$

**Lemma 25.** *The DG discretization (44) is*

a) *consistent, i.e.,*

$$\begin{aligned} \big(A_h\mathbf{y}, \mathbf{z}_h\big)_\Omega &= \big(A\mathbf{y}, \mathbf{z}_h\big)_\Omega\,, & \mathbf{y}\in Y_h\cap\mathcal{D}(A)\,,\ \mathbf{z}_h\in Y_h\,,\\ \big(A_h\mathbf{y}_h, \mathbf{z}\big)_\Omega &= -\big(\mathbf{y}_h, A\mathbf{z}\big)_\Omega\,, & \mathbf{y}_h\in Y_h\,,\ \mathbf{z}\in Y_h\cap\mathcal{D}(A)\,; \end{aligned}$$

b) *monotone / dissipative satisfying*

$$\big(A_h\mathbf{y}_h, \mathbf{y}_h\big)_\Omega = \frac{1}{2}\sum_{K\in\mathcal{K}_h}\sum_{F\in\mathcal{F}_K}\frac{1}{Z_K + Z_{K_F}}\bigg(\big\|[p_h]_{K,F}\big\|_F^2 + Z_K Z_{K_F}\big\|\mathbf{n}_K\cdot[\mathbf{v}_h]_{K,F}\big\|_F^2\bigg) \ge 0\,, \qquad \mathbf{y}_h\in Y_h\,.$$

*Proof.* It is sufficient to consider $r = 0$. For $\mathbf{y} = (\mathbf{v}, p) \in Y_h \cap \mathcal{D}(A)$ we obtain $\mathbf{n}_K \cdot [\mathbf{v}]_{K,F} = [p]_{K,F} = 0$ for $F \subset \mathcal{F}_K \in \Omega$, $p_h = 0$ on $F \in \mathcal{F}_K \cap \Gamma_\mathrm{V}$, and $\mathbf{n}_K \cdot \mathbf{v} = 0$ on $F \in \mathcal{F}_K \cap \Gamma_\mathrm{S}$, so that consistency is obtained by

$$\big(A_h\mathbf{y}, \mathbf{z}_h\big)_\Omega = \sum_{K\in\mathcal{K}_h} \big(A_{h,K}\mathbf{y}_K, \mathbf{z}_{h,K}\big)_K = \sum_{K\in\mathcal{K}_h} \big(A\mathbf{y}_K, \mathbf{z}_{h,K}\big)_K = \big(A\mathbf{y}, \mathbf{z}_h\big)_\Omega$$

for $\mathbf{z}_h \in Y_h$ since all flux terms on the faces are vanishing.
Integration by parts for $\mathbf{y}_h = (\mathbf{v}_h, p_h) \in Y_h$ and $\mathbf{z} = (\mathbf{w}, q) \in Y_h \cap \mathcal{D}(A)$ yields dual consistency by

$$\begin{aligned}
\big(A_h\mathbf{y}_h, \mathbf{z}\big)_\Omega &= \sum_{K\in\mathcal{K}_h} \Bigg( -\big(\nabla\cdot\mathbf{v}_{h,K}, q_K\big)_K - \big(\nabla p_{h,K}, \mathbf{w}_K\big)_K \\
&\qquad - \sum_{F\in\mathcal{F}_K\cap\Omega} \frac{1}{Z_K + Z_{K_F}} \big([p_h]_{K,F} + Z_{K_F}\mathbf{n}_K\cdot[\mathbf{v}_h]_{K,F}, q_K + Z_K\mathbf{n}_K\cdot\mathbf{w}_K\big)_F \\
&\qquad + \sum_{F\in\mathcal{F}_K\cap\Gamma_\mathrm{S}} \big(p_{h,K}, \mathbf{n}_K\cdot\mathbf{w}_K\big)_F + \sum_{F\in\mathcal{F}_K\cap\Gamma_\mathrm{V}} \big(\mathbf{n}_K\cdot\mathbf{v}_{h,K}, q_K\big)_F \Bigg) \\
&= \sum_{K\in\mathcal{K}_h} \Bigg( \big(\mathbf{v}_{h,K}, \nabla q_K\big)_K + \big(p_{h,K}, \nabla\cdot\mathbf{w}_K\big)_K \\
&\qquad - \sum_{F\in\mathcal{F}_K\cap\Omega} \Big( \frac{1}{Z_K + Z_{K_F}} \big([p_h]_{K,F} + Z_{K_F}\mathbf{n}_K\cdot[\mathbf{v}_h]_{K,F}, q_K + Z_K\mathbf{n}_K\cdot\mathbf{w}_K\big)_F \\
&\qquad\qquad + \big(p_{h,K}, \mathbf{n}_K\cdot\mathbf{w}_K\big)_F + \big(\mathbf{n}_K\cdot\mathbf{v}_{h,K}, q_K\big)_F \Big) \Bigg) \\
&= \sum_{K\in\mathcal{K}_h} \Bigg( -\big(\mathbf{y}_{h,K}, A\mathbf{z}_K\big)_K \\
&\qquad + \sum_{F\in\mathcal{F}_K\cap\Omega} \frac{1}{Z_K + Z_{K_F}} \Big( \big([p_h]_{K,F}, q_K\big)_F + Z_K Z_{K_F}\big(\mathbf{n}_K\cdot[\mathbf{v}_h]_{K,F}, \mathbf{n}_K\cdot\mathbf{w}_K\big)_F \Big) \Bigg) \\
&= -\big(\mathbf{y}_h, A\mathbf{z}\big)_\Omega \,.
\end{aligned}$$

For $\mathbf{y}_h = (\mathbf{v}_h, p_h) \in Y_h$ we obtain the identity

$$\begin{aligned}
&\big(A_h\mathbf{y}_h, \mathbf{y}_h\big)_\Omega = \sum_{K\in\mathcal{K}_h} \big(A_{h,K}\mathbf{y}_h, \mathbf{y}_h\big)_K \\
&= \sum_{K\in\mathcal{K}_h} \Bigg( -\big(\nabla\cdot\mathbf{v}_{h,K}, p_{h,K}\big)_K - \big(\nabla p_{h,K}, \mathbf{v}_{h,K}\big)_K \\
&\quad - \sum_{F\in\mathcal{F}_K\cap\Omega} \frac{1}{Z_K + Z_{K_F}} \big([p_h]_{K,F} + Z_{K_F}\mathbf{n}_K\cdot[\mathbf{v}_h]_{K,F}, p_{h,K} + Z_K\mathbf{n}_K\cdot\mathbf{v}_{h,K}\big)_F \\
&\quad + \sum_{F\in\mathcal{F}_K\cap\Gamma_\mathrm{S}} \frac{1}{Z_K} \big(p_{h,K}, p_{h,K} + Z_K\mathbf{n}_K\cdot\mathbf{v}_{h,K}\big)_F + \sum_{F\in\mathcal{F}_K\cap\Gamma_\mathrm{V}} \big(\mathbf{n}_K\cdot\mathbf{v}_{h,K}, p_{h,K} + Z_K\mathbf{n}_K\cdot\mathbf{v}_{h,K}\big)_F \Bigg) \\
&= \frac{1}{2} \sum_{K\in\mathcal{K}_h} \sum_{F\in\mathcal{F}_K} \frac{1}{Z_K + Z_{K_F}} \Big( \big\|[p_h]_{K,F}\big\|_F^2 + Z_K Z_{K_F} \big\|\mathbf{n}_K\cdot[\mathbf{v}_h]_{K,F}\big\|_F^2 \Big)
\end{aligned}$$

since we obtain, using $\big(\nabla\cdot\mathbf{v}_{h,K}, p_{h,K}\big)_K + \big(\nabla p_{h,K}, \mathbf{v}_{h,K}\big)_K = \big(\mathbf{n}_K\cdot\mathbf{v}_{h,K}, p_{h,K}\big)_{\partial K}$, for the remaining terms

$$\begin{aligned}
&\sum_{K\in\mathcal{K}_h} \Bigg( -\big(\mathbf{n}_K\cdot\mathbf{v}_{h,K}, p_{h,K}\big)_{\partial K} \\
&\quad - \sum_{F\in\mathcal{F}_K\cap\Omega} \frac{1}{Z_K + Z_{K_F}} \Big( Z_K\big([p_h]_{K,F}, \mathbf{n}_K\cdot\mathbf{v}_{h,K}\big)_F + Z_{K_F}\big(\mathbf{n}_K\cdot[\mathbf{v}_h]_{K,F}, p_{h,K}\big)_F \Big) \\
&\quad + \sum_{F\in\mathcal{F}_K\cap\Gamma_\mathrm{V}} \big(p_{h,K}, \mathbf{n}_K\cdot\mathbf{v}_{h,K}\big)_F + \sum_{F\in\mathcal{F}_K\cap\Gamma_\mathrm{S}} \big(\mathbf{n}_K\cdot\mathbf{v}_{h,K}, p_{h,K}\big)_F \Bigg) = 0\,.
\end{aligned}$$

$\square$

**The DG operator for visco-elasticity.** The operator $A_h = \sum_{K\in\mathcal{K}_h} A_{h,K} \in \mathcal{L}\big(Y_h, Y_h\big)$ with full upwind is defined by

$$\begin{aligned}
\big(A_{h,K}\mathbf{y}_h, \mathbf{z}_h\big)_K = &-\big(\nabla\cdot\boldsymbol{\sigma}_{h.K}, \boldsymbol{\psi}_{h,K}\big)_K - \big(\varepsilon(\mathbf{v}_{h,K}), \boldsymbol{\eta}_{h,K}\big)_K \\
&-\sum_{F\in\mathcal{F}_K} \frac{1}{Z_K^{\mathrm{p}} + Z_{K_F}^{\mathrm{p}}} \Big(\mathbf{n}_K\cdot\big([\boldsymbol{\sigma}_h]_{K,F}\mathbf{n}_K + Z_{K_F}^{\mathrm{p}}[\mathbf{v}_h]_{K,F}\big), \mathbf{n}_K\cdot\big(\boldsymbol{\eta}_{h,K}\mathbf{n}_K + Z_K^{\mathrm{p}}\mathbf{w}_{h,K}\big)\Big)_F \\
&-\sum_{F\in\mathcal{F}_K} \frac{1}{Z_K^{\mathrm{s}} + Z_{K_F}^{\mathrm{s}}} \Big(\mathbf{n}_K\times\big([\boldsymbol{\sigma}_h]_{K,F}\mathbf{n}_K + Z_{K_F}^{\mathrm{s}}[\mathbf{v}_h]_{K,F}\big), \mathbf{n}_K\times\big(\boldsymbol{\eta}_{h,K}\mathbf{n}_K + Z_K^{\mathrm{s}}\mathbf{w}_{h,K}\big)\Big)_F
\end{aligned}$$

for $\mathbf{y}_h = (\mathbf{v}_h, \boldsymbol{\sigma}_{0,h}, \dots, \boldsymbol{\sigma}_{r,h})$, $\mathbf{z}_h = (\mathbf{w}_h, \boldsymbol{\eta}_{0,h}, \dots, \boldsymbol{\eta}_{r,h}) \in Y_h$, $\boldsymbol{\sigma}_h = \sum \boldsymbol{\sigma}_{j,h}$, and $\boldsymbol{\eta}_h = \sum \boldsymbol{\eta}_{j,h}$. The coefficients $Z_K^{\mathrm{p}} = \sqrt{(2\mu+\lambda)\rho}\,|_K$ and $Z_K^{\mathrm{s}} = \sqrt{\mu\rho}\,|_K$ are the impedance of compressional waves and shear waves, respectively. On boundary faces $F \in \mathcal{F}_h \cap \Gamma_{\mathrm{V}}$, we set $[\mathbf{v}_h]_{K,F} = -2\mathbf{v}_h$ and $[\boldsymbol{\sigma}_h]_{K,F}\mathbf{n}_K = \mathbf{0}$, and on $F \in \mathcal{F}_h \cap \Gamma_{\mathrm{S}}$ we set $[\mathbf{v}_h]_{K,F} = \mathbf{0}$ and $[\boldsymbol{\sigma}_h]_{K,F}\mathbf{n}_K = -2\,\boldsymbol{\sigma}_h\mathbf{n}_K$. We have

$$\begin{aligned}
\big(A_h\mathbf{y}_h, \mathbf{y}_h\big)_\Omega = \frac{1}{2}\sum_{K\in\mathcal{K}_h}\sum_{F\in\mathcal{F}_K} \Bigg( &\frac{\big\|\mathbf{n}_K\cdot\big([\boldsymbol{\sigma}_h]_{K,F}\mathbf{n}_K\big)\big\|_F^2 + Z_K^{\mathrm{p}} Z_{K_F}^{\mathrm{p}} \big\|\mathbf{n}_K\cdot[\mathbf{v}_h]_{K,F}\big\|_F^2}{Z_K^{\mathrm{p}} + Z_{K_F}^{\mathrm{p}}} \\
&+ \frac{\big\|\mathbf{n}_K\times\big([\boldsymbol{\sigma}_h]_{K,F}\mathbf{n}_K\big)\big\|_F^2 + Z_K^{\mathrm{s}} Z_{K_F}^{\mathrm{s}} \big\|\mathbf{n}_K\times[\mathbf{v}_h]_{K,F}\big\|_F^2}{Z_K^{\mathrm{s}} + Z_{K_F}^{\mathrm{s}}}\Bigg) \geq 0\,.
\end{aligned}$$

**The DG operator for linear electro-magnetic waves.** For $(\mathbf{H}_h, \mathbf{E}_h), (\boldsymbol{\varphi}_h, \boldsymbol{\psi}_h) \in Y_h$ we have

$$\begin{aligned}
\big(A_h(\mathbf{H}_h, \mathbf{E}_h), (\boldsymbol{\varphi}_{h,K}, \boldsymbol{\psi}_{h,K})\big)_{0,K} = &(\operatorname{curl}\mathbf{E}_{h,K}, \boldsymbol{\varphi}_{h,K})_{0,K} - \big(\operatorname{curl}\mathbf{H}_{h,K}, \boldsymbol{\psi}_{h,K}\big)_{0,K} \\
&- \sum_{F\in\mathcal{F}_K} \frac{1}{Z_K + Z_{K_F}} \Big(\big(Z_{K_F}[\mathbf{E}_h]_{K,F} + \mathbf{n}_K\times[\mathbf{H}_h]_{K,F}, \mathbf{n}_K\times\boldsymbol{\varphi}_{h,K}\big)_F \\
&\qquad + \big(Z_{K_F}\mathbf{n}_K\times[\mathbf{E}_h]_{K,F} - [\mathbf{H}_h]_{K,F}, Z_K\mathbf{n}_K\times\boldsymbol{\psi}_{h,K}\big)_F\Big) \\
&+ \sum_{F\in\mathcal{F}_K\cap\Gamma_{\mathrm{I}}} \big(\zeta\,\mathbf{n}_K\times\mathbf{E}_{K,h}, \mathbf{n}_K\times\boldsymbol{\varphi}_{h,K}\big)_F
\end{aligned}$$

with coefficient $Z_K = \sqrt{\varepsilon_K/\mu_K}$ and impedance $\zeta$.
On boundary faces $F \in \mathcal{F}_h \cap \Gamma_{\mathrm{E}}$, perfect conducting boundary conditions are modeled by the (only virtual) definition of $\mathbf{n}_K\times\mathbf{E}_h = -\mathbf{n}_K\times\mathbf{E}_h$ and $\mathbf{n}_K\times\mathbf{H}_h = \mathbf{n}_K\times\mathbf{H}_h$, i.e., $\mathbf{n}_K\times[\mathbf{E}]_{K,F} = -2\mathbf{n}_K\times\mathbf{E}_h$ and $\mathbf{n}_K\times[\mathbf{H}_h]_{K,F} = \mathbf{0}$. On impedance boundary faces $F \in \mathcal{F}_h \cap \Gamma_{\mathrm{I}}$, we set $\mathbf{n}_K\times[\mathbf{E}]_{K,F} = \mathbf{0}$ and $\mathbf{n}_K\times[\mathbf{H}_h]_{K,F} = -2\mathbf{n}_K\times\mathbf{H}_h$. With the same arguments as for the acoustic case we obtain

$$\begin{aligned}
&\big(A_h(\mathbf{H}_h, \mathbf{E}_h), (\mathbf{H}_h, \mathbf{E}_h)\big)_{0,\Omega} \\
&\quad = \frac{1}{2}\sum_{K\in\mathcal{K}_h}\sum_{F\in\mathcal{F}_K} \frac{Z_K Z_{K_F}\big\|\mathbf{n}_K\times[\mathbf{E}_h]_{K,F}\big\|_F^2 + \big\|\mathbf{n}_K\times[\mathbf{H}_h]_{K,F}\big\|_F^2}{Z_K + Z_{K_F}} \\
&\qquad + \sum_{F\in\mathcal{F}_h\cap\Gamma_{\mathrm{I}}} \zeta\,\big\|\mathbf{n}_K\times\mathbf{E}_h\big\|_F^2 \geq 0\,.
\end{aligned}$$

**Bibliographic comments.** An introduction to discontinuous Galerkin methods for hyperbolic conservation laws is given, e.g., in [Hesthaven and Warburton, 2008,Hesthaven, 2017]. The numerical flux for wave equations is evaluated in [Hochbruck et al., 2015] and extended to viscous waves in [Ziegler, 2019]. For the explicit evaluation of the numerical flux for inhomogeneous boundary conditions we refer to [Dörfler et al., 2019].

# 4. A Petrov–Galerkin space-time approximation for linear hyperbolic systems

We introduce and analyze a variational discretization in space and time by extending the discontinuous Galerkin method in space to a Petrov–Galerkin method in time and space. We verify discrete inf-sup stability, and this yields well-posedness, stability and convergence for strong solutions. By duality, this extends to convergence for weak solutions. Finally, we address a posteriori error bounds. For a given error functional, the corresponding dual solution is computed and an error indicator is defined by weighted residuals. A reliable error estimator of weak solutions is obtained computing local conforming reconstructions.

## 4.1. Decomposition of the space-time cylinder

For the discretization, we use tensor product space-time cells combining the mesh in space (see Sect. 3.6) with a decomposition in time. For $0 = t_0 < t_1 < \cdots < t_N = T$, we define

$$I_h = (t_0, t_1) \cup \cdots \cup (t_{N-1}, t_N) \subset I = (0, T)\,.$$

Together with the decomposition in space $\Omega_h = \bigcup_{K\in\mathcal{K}_h} K$ into open cells $K \subset \Omega \subset \mathbb{R}^d$ we obtain a decomposition $Q_h = I_h \times \Omega_h = \bigcup_{R\in\mathcal{R}_h} R$ of the space-time cylinder $Q = I \times \Omega \subset \mathbb{R}^{1+d}$, so that $\overline{Q} = Q_h \cup \partial Q_h$, where $\partial Q_h$ is the space-time skeleton.

For every space-time cell $R = (t_{n-1}, t_n) \times K$ we select polynomial degrees $p_R = p_{n,K} \geq 1$ in time and $q_R = q_{n,K} \geq 0$ in space. This defines the discontinuous test space in the space-time cylinder

$$W_h = \prod_{R\in\mathcal{R}_h} \mathbb{P}_{p_R-1} \otimes \mathbb{P}_{q_R}(K;\mathbb{R}^m) \subset \mathbb{P}(I_h \times \Omega_h;\mathbb{R}^m) = \prod_{R\in\mathcal{R}_h} \mathbb{P}(R;\mathbb{R}^m) \subset \mathrm{L}_2(Q;\mathbb{R}^m)\,,$$

where $\mathbb{P}_p$ are the polynomials up to order $p$, $\mathbb{P}_q(K)$ are the polynomials up to order $q$ in $K$, and $\mathbb{P}(R)$ are polynomials of any degree in $R$.

Defining the discontinuous spaces

$$Y_{n,h} = \prod_{K\in\mathcal{K}_h} \mathbb{P}_{q_{n,K}}(K;\mathbb{R}^m) \subset \mathbb{P}(\Omega_h;\mathbb{R}^m) \subset \mathrm{L}_2(\Omega;\mathbb{R}^m)\,, \quad Y_h = Y_{1,h} + \cdots + Y_{N,h}\,,$$

we observe $W_h \subset \mathrm{L}_2(0,T;Y_h)$, and in every time slice $\mathbf{w}_h(t) \in Y_{n,h} \subset Y_h$ for all $t \in (t_{n-1}, t_n)$ and $\mathbf{w}_h \in W_h$.

## 4.2. The Petrov–Galerkin setting

Let $L_h = M_h\partial_t + D_h + A_h \colon \mathrm{H}^1(0,T;Y_h) \longrightarrow \mathrm{L}_2(0,T;Y_h)$ be the linear mapping approximating the differential operator $L = M\partial_t + D + A$ with the following properties:

a) $M_h \in \mathcal{L}(Y_h, Y_h)$ is uniformly positive definite, i.e., $c_M > 0$ exists with

$$(M_h\mathbf{y}_h, \mathbf{y}_h)_\Omega \geq c_M \|\mathbf{y}_h\|_W^2\,, \qquad \mathbf{y}_h \in Y_h\,; \tag{46a}$$

b) $D_h \in \mathcal{L}(Y_h, Y_h)$ is monotone, i.e.,

$$(D_h\mathbf{y}_h, \mathbf{y}_h)_\Omega \geq 0\,, \qquad \mathbf{y}_h \in Y_h\,; \tag{46b}$$

c) $A_h \in \mathcal{L}(Y_h, Y_h)$ is monotone and consistent, i.e.,

$$(A_h\mathbf{y}_h, \mathbf{y}_h)_\Omega \geq 0\,, \qquad \mathbf{y}_h \in Y_h\,, \tag{46c}$$

$$\begin{aligned} (A_h\mathbf{z}, \mathbf{y}_h)_\Omega &= (A\mathbf{z}, \mathbf{y}_h)_\Omega\,,\\ (A_h\mathbf{y}_h, \mathbf{z})_\Omega &= -(\mathbf{y}_h, A\mathbf{z})_\Omega\,, \qquad \mathbf{z} \in Y_h \cap \mathcal{D}(A)\,. \end{aligned} \tag{46d}$$

The operators $M_h, D_h, A_h$ do not depend on the time variable $t \in (0,T)$, i.e., they are defined in $Y_h \subset \mathbb{P}(\Omega_h;\mathbb{R}^m)$, but the operators do not depend on the chosen local ansatz and test spaces.

In the next step we construct a suitable ansatz space $V_h \subset \mathbb{P}(Q_h;\mathbb{R}^m)$. In every time slice $(t_{n-1}, t_n)$ let

$$\Pi_{n,h} \colon \mathrm{L}_2(\Omega;\mathbb{R}^m) \longrightarrow Y_{n,h}$$

be the weighted $\mathrm{L}_2$-projection defined by

$$\big(M_h\Pi_{n,h}\mathbf{y}, \mathbf{z}_h\big)_\Omega = \big(M_h\mathbf{y}, \mathbf{z}_h\big)_\Omega\,, \qquad \mathbf{y} \in \mathrm{L}_2(\Omega;\mathbb{R}^m)\,,\ \mathbf{z}_h \in Y_{n,h}$$

corresponding to the norm

$$\|\mathbf{y}_h\|_{Y_h} = \sqrt{\big(M_h\mathbf{y}_h, \mathbf{y}_h\big)_\Omega}\,, \qquad \mathbf{y}_h \in Y_h$$

For $\mathbf{v}_h \in \mathbb{P}(I_h \times \Omega_h; \mathbb{R}^m)$ let $\mathbf{v}_{n,h} \in \mathbb{P}([t_{n-1}, t_n] \times \Omega_h; \mathbb{R}^m)$ be the extension of $\mathbf{v}_h|_{(t_{n-1},t_n)\times\Omega_h}$ to $[t_{n-1}, t_n]$. Then, we define

$$\begin{aligned} V_h = \Big\{\mathbf{v}_h \in \prod_{R=(t_{n-1},t_n)\times K\in\mathcal{R}_h} \mathbb{P}_{p_R} \otimes \mathbb{P}_{q_R}(K; \mathbb{R}^m) \subset \mathbb{P}(I_h \times \Omega_h; \mathbb{R}^m) \colon \\ \mathbf{v}_h(0) = \mathbf{0}\,,\ \mathbf{v}_{n,h}(t_{n-1}) = \Pi_{n,h}\mathbf{v}_{n-1,h}(t_{n-1}) \text{ for } n = 2, \dots, N\Big\} \subset \mathrm{H}^1(0, T; Y_h)\,. \end{aligned}$$

By construction, we have $\partial_t V_h = W_h$ in $I_h$ and $\dim V_h = \dim W_h$. Note that $V_h$ includes homogeneous initial data. For inhomogeneous initial data $\mathbf{u}_0$ we define the affine space

$$\begin{aligned} V_h(\mathbf{u}_0) = \Big\{\mathbf{v}_h \in \prod_{R=(t_{n-1},t_n)\times K\in\mathcal{R}_h} \mathbb{P}_{p_R} \otimes \mathbb{P}_{q_R}(K; \mathbb{R}^m) \subset \mathbb{P}(I_h \times \Omega_h; \mathbb{R}^m) \colon \\ \mathbf{v}_h(0) = \Pi_{1,h}\mathbf{u}_0 \text{ for } t = 0\,,\ \mathbf{v}_{n,h}(t_{n-1}) = \Pi_{n,h}\mathbf{v}_{n-1,h}(t_{n-1}) \text{ for } n = 2, \dots, N\Big\} \subset \mathrm{H}^1(0, T; Y_h)\,. \end{aligned} \tag{47}$$

### 4.3. **Inf-sup stability**

Let

$$\Pi_h \colon \mathrm{L}_2(Q; \mathbb{R}^m) \longrightarrow W_h$$

be the projection defined by

$$\big(M_h\Pi_h\mathbf{v}, \mathbf{w}_h\big)_Q = \big(M_h\mathbf{v}, \mathbf{w}_h\big)_Q\,, \qquad \mathbf{v} \in \mathrm{L}_2(Q; \mathbb{R}^m)\,,\ \mathbf{w}_h \in W_h\,.$$

Note that $\Pi_h M_h\mathbf{v}_h = M_h\Pi_h\mathbf{v}_h$ and $\Pi_h A_h\mathbf{v}_h = A_h\Pi_h\mathbf{v}_h$ for $\mathbf{v}_h \in \mathrm{L}_2(0, T; Y_h)$.
The analysis of the discretization is based on the norms

$$\|\mathbf{w}_h\|_{W_h} = \sqrt{\big(M_h\mathbf{w}_h, \mathbf{w}_h\big)_Q}\,, \quad \|\mathbf{f}_h\|_{W_h^*} = \sqrt{\big(M_h^{-1}\mathbf{f}_h, \mathbf{f}_h\big)_Q}\,, \quad \mathbf{w}_h\,, \mathbf{f}_h \in \mathrm{L}_2(Q; \mathbb{R}^m)$$

and

$$\|\mathbf{v}_h\|_{V_h} = \sqrt{\|\mathbf{v}_h\|^2_{W_h} + \|\Pi_h M_h^{-1} L_h\mathbf{v}_h\|^2_{W_h}}\,, \qquad \mathbf{v}_h \in \mathrm{H}^1(0, T; Y_h)\,. \tag{48}$$

**Theorem 26.** *The bilinear form $b_h \colon \mathrm{H}^1(0, T; Y_h) \times \mathrm{L}_2(Q; \mathbb{R}^m) \longrightarrow \mathbb{R}$ defined by $b_h(\mathbf{v}_h, \mathbf{w}_h) = (L_h\mathbf{v}_h, \mathbf{w}_h)_Q$ is inf-sup stable in $V_h \times W_h$ satisfying*

$$\sup_{\mathbf{w}_h\in W_h\setminus\{\mathbf{0}\}} \frac{b_h(\mathbf{v}_h, \mathbf{w}_h)}{\|\mathbf{w}_h\|_{W_h}} \geq \beta\,\|\mathbf{v}_h\|_{V_h}\,, \quad \mathbf{v}_h \in V_h \quad \text{with } \beta = \frac{1}{\sqrt{4T^2 + 1}}\,.$$

*Thus, for given $\mathbf{f} \in \mathrm{L}_2(Q; \mathbb{R}^m)$, a unique solution $\mathbf{u}_h \in V_h$ of*

$$(L_h\mathbf{u}_h, \mathbf{w}_h)_Q = (\mathbf{f}, \mathbf{w}_h)_Q\,, \qquad \mathbf{w}_h \in W_h, \tag{49}$$

*exists satisfying the a priori bound $\|\mathbf{u}_h\|_{V_h} \leq \beta^{-1}\|\Pi_h\mathbf{f}\|_{W_h^*}$.*

The stability constant $\beta > 0$ is the same as in the continuous case in Thm. 19.
The proof of the inf-sup stability is based on the following estimates.

**Lemma 27.** *Let $\lambda_{n,k} \in \mathbb{P}_k$, $k = 0, 1, 2, \dots$, be the orthonormal Legendre polynomials in $\mathrm{L}_2(t_{n-1}, t_n)$. Then, we have $\big(t\partial_t\lambda_{n,k}, \lambda_{n,k}\big)_{(t_{n-1},t_n)} \geq 0$.*

*Proof.* The orthonormal Legendre polynomials $\lambda_{n,k}$ with respect to $\big(\cdot,\cdot\big)_{(t_{n-1},t_n)}$ are given by scaling the orthogonal polynomials $\tilde\lambda_{n,k}$

$$\lambda_{n,k}(t) = c_{n,k}\tilde\lambda_{n,k}(t)\,,\ \ \tilde\lambda_{n,k}(t) = \partial_t^k\big((t - t_{n-1})(t - t_n)\big)^k\,,\ \ c_{n,k} = \|\tilde\lambda_{n,k}\|^{-1}_{(t_{n-1},t_n)}\,.$$

For $k=0$ we have $\partial_t \lambda_{n,0} = 0$ and thus $\big(t\partial_t \lambda_{n,0}, \lambda_{n,0}\big)_{(t_{n-1},t_n)} = 0$. For $k \geq 1$ we have

$$\begin{aligned}
\big(t\partial_t \lambda_{n,k}, \lambda_{n,k}\big)_{(t_{n-1},t_n)} &= \Big(tc_{n,k}\partial_t^{k+1}\big((t-t_{n-1})(t-t_n)\big)^k, \lambda_{n,k}\Big)_{(t_{n-1},t_n)} \\
&= \big(tc_{n,k}\partial_t^{k+1} t^{2k}, \lambda_{n,k}\big)_{(t_{n-1},t_n)} \\
&= \big(c_{n,k}k\partial_t^{k} t^{2k}, \lambda_{n,k}\big)_{(t_{n-1},t_n)} = k\big(\lambda_{n,k}, \lambda_{n,k}\big)_{(t_{n-1},t_n)} = k > 0
\end{aligned}$$

using $t\,\partial_t^{k+1} t^{2k} = t\Big(2k(2k-1)\cdots(k+1)k\,t^{k-1}\Big) = k\,\partial_t^k t^{2k}$. $\square$

**Lemma 28.** *Let $\underline{X} = (X_{kn}) \in \mathbb{R}^{N\times N}_{\mathrm{sym}}$ be a symmetric and positive semidefinite matrix, and $\underline{Y} = (Y_{kn}) \in \mathbb{R}^{N\times N}$ be a positive semidefinite matrix. Then,*

$$\underline{X} : \underline{Y} = \sum_{k,m=1}^{N} X_{km} Y_{km} \geq 0\,.$$

*Proof.* Let $(\mu_n, \underline{w}_n)$, $n = 1,\dots,N$ be a complete eigensystem of $\underline{X}$ with $\mu_n \geq 0$ and $\underline{w}_n = (w_{nk})_{k=1,\dots,N} \in \mathbb{R}^N$, so that $\underline{X} = \sum_{n=1}^N \mu_n \underline{w}_n \underline{w}_n^\top$. Then, we have

$$\underline{X} : \underline{Y} = \sum_{k,m=1}^{N} X_{km} Y_{km} = \sum_{k,m,n=1}^{N} \mu_n w_{nk} w_{nm} Y_{km} = \sum_{n=1}^{N} \mu_n \underline{w}_n^\top \underline{Y}\,\underline{w}_n \geq 0\,.$$

$\square$

**Lemma 29.** *We have for $\mathbf{v}_h \in V_h$*

$$\|\mathbf{v}_h\|_{W_h} \leq 2T\,\|\Pi_h M_h^{-1} L_h \mathbf{v}_h\|_{W_h}\,, \qquad \mathbf{v}_h \in V_h\,. \tag{50}$$

This shows that Lem. 13 extends to the discrete estimate also with $C_L = 2T$.

*Proof.* Set $p = \max\limits_{R\in\mathcal{R}_h} p_R$. For $\mathbf{v}_h \in V_h$ in every time slice $(t_{n-1}, t_n)$ a representation

$$\mathbf{v}_{n,h}(t,\mathbf{x}) = \sum_{k=0}^{p} \lambda_{n,k}(t)\mathbf{v}_{n,k,h}(\mathbf{x})\,, \qquad \mathbf{v}_{n,k,h} \in Y_{n,h}\,,\ (t,\mathbf{x}) \in (t_{n-1},t_n)\times\Omega_h$$

exists with $\mathbf{v}_{n,k,h}(\mathbf{x}) = \mathbf{0}$ for $(t,\mathbf{x}) \in R = (t_{n-1},t_n)\times K$ and $k > p_R$, so that

$$\Pi_h \mathbf{v}_{n,h}(t,\mathbf{x}) = \sum_{k=0}^{p_R-1} \lambda_{n,k}(t)\mathbf{v}_{n,k,h}(\mathbf{x}) = \sum_{k=0}^{p} \lambda_{n,k}(t)\hat{\mathbf{v}}_{n,k,h}(\mathbf{x})\,, \qquad (t,\mathbf{x}) \in R = (t_{n-1},t_n)\times K$$

with $\hat{\mathbf{v}}_{n,k,h}(\mathbf{x}) = \mathbf{v}_{n,k,h}(\mathbf{x})$ for $k < p_R$ and $\hat{\mathbf{v}}_{n,k,h}(\mathbf{x}) = \mathbf{0}$ for $k \geq p_R$.
The proof of (50) relies on the application of Fubini's theorem

$$\int_0^T \int_0^t \phi(s)\,\mathrm{d}s\,\mathrm{d}t = \int_0^T d_T(t)\phi(t)\,\mathrm{d}t\,, \qquad \phi \in \mathrm{L}_1(0,T) \tag{51}$$

and on estimates with respect to the weighting function in time $d_T(t) = T - t$.

In the first step, we show

$$\big(M_h\partial_t \mathbf{v}_h, d_T\mathbf{v}_h\big)_Q \le \big(M_h\partial_t \mathbf{v}_h, d_T\Pi_h\mathbf{v}_h\big)_Q\,, \tag{52}$$

$$0 \le \big(\Pi_h A_h\mathbf{v}_h, d_T\Pi_h\mathbf{v}_h\big)_Q\,, \tag{53}$$

$$0 \le \big(\Pi_h D_h\mathbf{v}_h, d_T\Pi_h\mathbf{v}_h\big)_Q\,. \tag{54}$$

Since $A_h$ and $D_h$ are monotone, we obtain (53) and (54) from Lem. 28 applied to

$$\begin{aligned}
\big(\Pi_h A_h\mathbf{v}_h, d_T\Pi_h\mathbf{v}_h\big)_Q &= \sum_{n=1}^N \big(\Pi_h A_h\mathbf{v}_{n,h}, d_T\Pi_h\mathbf{v}_{n,h}\big)_{(t_{n-1},t_n)\times\Omega}\\
&= \sum_{n=1}^N \sum_{R=(t_{n-1},t_n)\times K} \sum_{k=0}^{p_R-1}\sum_{l=0}^{p_R-1} \big(\lambda_{n,k}, d_T\lambda_{n,l}\big)_{(t_{n-1},t_n)} \big(A_h\mathbf{v}_{n,k,h}, \mathbf{v}_{n,l,h}\big)_K\\
&= \sum_{n=1}^N \sum_{k=0}^p\sum_{l=0}^p \big(\lambda_{n,k}, d_T\lambda_{n,l}\big)_{(t_{n-1},t_n)} \big(A_h\hat{\mathbf{v}}_{n,k,h}, \hat{\mathbf{v}}_{n,l,h}\big)_\Omega \ge 0\,,\\
\big(\Pi_h D_h\mathbf{v}_h, d_T\Pi_h\mathbf{v}_h\big)_Q &= \sum_{n=1}^N \sum_{R=(t_{n-1},t_n)\times K} \sum_{k=0}^{p_R-1}\sum_{l=0}^{p_R-1} \big(\lambda_{n,k}, d_T\lambda_{n,l}\big)_{(t_{n-1},t_n)} \big(D_h\mathbf{v}_{n,k,h}, \mathbf{v}_{n,l,h}\big)_K\\
&= \sum_{n=1}^N \sum_{k=0}^p\sum_{l=0}^p \big(\lambda_{n,k}, d_T\lambda_{n,l}\big)_{(t_{n-1},t_n)} \big(D_h\hat{\mathbf{v}}_{n,k,h}, \hat{\mathbf{v}}_{n,l,h}\big)_\Omega \ge 0\,.
\end{aligned}$$

For $k\ge 1$ we have $\big(d_T\partial_t\lambda_{n,k},\lambda_{n,k}\big)_{(t_{n-1},t_n)} = -\big(t\partial_t\lambda_{n,k},\lambda_{n,k}\big)_{(t_{n-1},t_n)} < 0$ by Lem. 27, which gives

$$\begin{aligned}
\big(d_T M_h\partial_t\mathbf{v}_h, \mathbf{v}_h - \Pi_h\mathbf{v}_h\big)_Q &= \sum_{n=1}^N \big(d_T M_h\partial_t\mathbf{v}_h, \mathbf{v}_h - \Pi_h\mathbf{v}_h\big)_{(t_{n-1},t_n)\times\Omega}\\
&= \sum_{n=1}^N \sum_{R=(t_{n-1},t_n)\times K} \sum_{k=0}^{p_R} \big(d_T\partial_t\lambda_{n,k}, \lambda_{n,p_R}\big)_{(t_{n-1},t_n)} \big(M_h\mathbf{v}_{n,k,h}, \mathbf{v}_{n,p_R,h}\big)_K\\
&= \sum_{n=1}^N \sum_{R=(t_{n-1},t_n)\times K} \big(d_T\partial_t\lambda_{n,p_R}, \lambda_{n,p_R}\big)_{(t_{n-1},t_n)} \big(M_h\mathbf{v}_{n,p_R,h}, \mathbf{v}_{n,p_R,h}\big)_K \le 0\,.
\end{aligned}$$

Thus we obtain (52) by

$$\big(M_h\partial_t\mathbf{v}_h, d_T\mathbf{v}_h\big)_Q = \big(d_T M_h\partial_t\mathbf{v}_h, \mathbf{v}_h\big)_Q \le \big(d_T M_h\partial_t\mathbf{v}_h, \Pi_h\mathbf{v}_h\big)_Q = \big(M_h\partial_t\mathbf{v}_h, d_T\Pi_h\mathbf{v}_h\big)_Q\,.$$

Finally, we show the assertion (50). We have for $k = 2,\dots,N$

$$\big\|\mathbf{v}_{k,h}(t_{k-1})\big\|_{Y_h} = \|\Pi_{k,h}\mathbf{v}_{k-1,h}(t_{k-1})\|_{Y_h} \le \|\mathbf{v}_{k-1,h}(t_{k-1})\|_{Y_h}\,,$$

so that for all $t\in(t_{n-1},t_n)$ using $\mathbf{v}_h(0) = \mathbf{v}_{1,h}(0) = \mathbf{0}$

$$\begin{aligned}
\big\|\mathbf{v}_h(t)\big\|_{Y_h}^2 &= \big\|\mathbf{v}_h(t)\big\|_{Y_h}^2 + \sum_{k=2}^n \Big(\|\Pi_{k,h}\mathbf{v}_{k-1,h}(t_{k-1})\|_{Y_h}^2 - \big\|\mathbf{v}_{k,h}(t_{k-1})\big\|_{Y_h}^2\Big) - \big\|\mathbf{v}_{1,h}(0)\big\|_{Y_h}^2\\
&\le \big\|\mathbf{v}_h(t)\big\|_{Y_h}^2 + \sum_{k=2}^n \Big(\|\mathbf{v}_{k-1,h}(t_{k-1})\|_{Y_h}^2 - \big\|\mathbf{v}_{k,h}(t_{k-1})\big\|_{Y_h}^2\Big) - \big\|\mathbf{v}_{1,h}(0)\big\|_{Y_h}^2\\
&= \big\|\mathbf{v}_h(t)\big\|_{Y_h}^2 - \big\|\mathbf{v}_{n,h}(t_{n-1})\big\|_{Y_h}^2 + \sum_{k=1}^{n-1} \Big(\big\|\mathbf{v}_{k,h}(t_k)\big\|_{Y_h}^2 - \big\|\mathbf{v}_{k,h}(t_{k-1})\big\|_{Y_h}^2\Big)\\
&= \int_{t_{n-1}}^t \partial_s\big(M_h\mathbf{v}_{n,h}(s), \mathbf{v}_{n,h}(s)\big)_\Omega\,\mathrm{d}s + \sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k} \partial_s\big(M_h\mathbf{v}_{n,h}(s), \mathbf{v}_{n,h}(s)\big)_\Omega\,\mathrm{d}s\\
&= 2\int_0^t \big(M_h\partial_s\mathbf{v}_h(s), \mathbf{v}_h(s)\big)_\Omega\,\mathrm{d}s
\end{aligned}$$

and thus using (51), (52), (53), and (54) we obtain (50) by

$$\begin{aligned}
\|\mathbf{v}_h\|_{W_h}^2 &= \int_0^T \big(M_h\mathbf{v}_h(t),\mathbf{v}_h(t)\big)_\Omega \,\mathrm{d}t \le 2\int_0^T\int_0^t \big(M_h\partial_s\mathbf{v}_h(s),\mathbf{v}_h(s)\big)_\Omega\,\mathrm{d}s\,\mathrm{d}t\\
&= 2\int_0^T d_T(t)\big(M_h\partial_t\mathbf{v}_h(t),\mathbf{v}_h(t)\big)_\Omega\,\mathrm{d}t = 2\big(M_h\partial_t\mathbf{v}_h, d_T\mathbf{v}_h\big)_Q\\
&\le 2\big(M_h\partial_t\mathbf{v}_h, d_T\Pi_h\mathbf{v}_h\big)_Q = 2\big(M_h\Pi_h\partial_t\mathbf{v}_h, d_T\Pi_h\mathbf{v}_h\big)_Q = 2\big(\Pi_h M_h\partial_t\mathbf{v}_h, d_T\Pi_h\mathbf{v}_h\big)_Q \le 2\big(\Pi_h L_h\mathbf{v}_h, d_T\Pi_h\mathbf{v}_h\big)_Q\\
&= 2\big(M_h^{-1}\Pi_h L_h\mathbf{v}_h, M_h d_T\Pi_h\mathbf{v}_h\big)_Q = 2\big(\Pi_h M_h^{-1}L_h\mathbf{v}_h, M_h d_T\Pi_h\mathbf{v}_h\big)_Q\\
&\le 2\,\|\Pi_h M_h^{-1}L_h\mathbf{v}_h\|_{W_h}\|d_T\Pi_h\mathbf{v}_h\|_{W_h} \le 2T\,\|\Pi_h M_h^{-1}L_h\mathbf{v}_h\|_{W_h}\|\mathbf{v}_h\|_{W_h}\,.
\end{aligned}$$

□

Now we can prove Thm. 26.

*Proof.* (Thm. 26) For $\mathbf{v}_h\in V_h\setminus\{\mathbf{0}\}$ we have

$$b_h(\mathbf{v}_h,\mathbf{w}_h) = \big(L_h\mathbf{v}_h,\mathbf{w}_h\big)_Q = \big(M_h^{-1}L_h\mathbf{v}_h,\mathbf{w}_h\big)_{W_h} = \big(\Pi_h M_h^{-1}L_h\mathbf{v}_h,\mathbf{w}_h\big)_{W_h}\,,$$

and we test with $\mathbf{w}_h = \Pi_h M_h^{-1}L_h\mathbf{v}_h$, so that

$$\sup_{\mathbf{w}_h\in W_h\setminus\{\mathbf{0}\}}\frac{b_h(\mathbf{v}_h,\mathbf{w}_h)}{\|\mathbf{w}_h\|_{W_h}} \ge \frac{b_h(\mathbf{v}_h,\Pi_h M_h^{-1}L_h\mathbf{v}_h)}{\|\Pi_h M_h^{-1}L_h\mathbf{v}_h\|_{W_h}} = \|\Pi_h M_h^{-1}L_h\mathbf{v}_h\|_{W_h} \ge (4T^2+1)^{-1/2}\|\mathbf{v}_h\|_{V_h}\,.$$

using $\|\mathbf{v}_h\|_{V_h}^2 = \|\mathbf{v}_h\|_{W_h}^2 + \|\Pi_h M_h^{-1}L_h\mathbf{v}_h\|_{W_h}^2 \le (4T^2+1)\|\Pi_h M_h^{-1}L_h\mathbf{v}_h\|_{W_h}^2$ inserting the estimate (50) in Lem. 29.

□

## 4.4. **Convergence for strong solutions**

For the error estimate with respect to the norm in $V_h$ we need to extend the norm $\|\cdot\|_{V_h}$ such that the error can be evaluated in this norm. For sufficiently smooth functions the operator $A_h$ can be extended by (40), so that $L_h$ and thus the norm in $V_h$ is well-defined.

**Theorem 30.** *Let $\mathbf{u}\in V$ be the strong solution of $L\mathbf{u}=\mathbf{f}$, and let $\mathbf{u}_h\in V_h$ be the approximation solving (49). If the solution is sufficiently smooth, we obtain the a priori error estimate*

$$\begin{aligned}
\|\mathbf{u}-\mathbf{u}_h\|_{V_h} \le C\big(\triangle t^p + \triangle x^q\big)&\Big(\|\partial_t^{p+1}\mathbf{u}\|_Q + \|\mathrm{D}^{q+1}\mathbf{u}\|_Q\Big)\\
&+\beta^{-1}\big\|M_h^{-1/2}(M_h-M)M^{-1/2}\big\|_\infty\|\partial_t\mathbf{u}\|_W\\
&+\beta^{-1}\big\|M_h^{-1/2}(D_h-D)M^{-1/2}\big\|_\infty\|\mathbf{u}\|_W
\end{aligned}$$

*for $\triangle t$, $\triangle x$ and $p,q\ge 1$ with $\triangle t\ge t_n-t_{n-1}$, $\triangle x\ge \mathrm{diam}(K)$, $p\le p_R$ and $q\le q_R$ (for all $n,K,R$), and with a constant $C>0$ depending on $\beta=(4T^2+1)^{-1/2}$, on the material parameters in $M$, and on the mesh regularity.*

*Proof.* For the solution we assume the regularity $\mathbf{u}\in \mathrm{H}^{p+1}\big(0,T;\mathrm{L}_2(\Omega;\mathbb{R}^m)\big)\cap \mathrm{L}_2\big(0,T;\mathrm{H}^{q+1}(\Omega;\mathbb{R}^m)\big)$, hence there exists an interpolant $\mathbf{v}_h\in V_h$ such that

$$\|\mathbf{u}-\mathbf{v}_h\|_{V_h} \le C\big(\triangle t^p+\triangle x^q\big)\Big(\|\partial_t^{p+1}\mathbf{u}\|_Q + \|\mathrm{D}^{q+1}\mathbf{u}\|_Q\Big)\,. \tag{55}$$

Moreover, $A_h\mathbf{u}$ is well-defined and consistent satisfying (41). We have

$$
\begin{aligned}
b_h(\mathbf{v}_h-\mathbf{u}_h,\mathbf{w}_h) &= b_h(\mathbf{v}_h,\mathbf{w}_h)-b_h(\mathbf{u}_h,\mathbf{w}_h)=b_h(\mathbf{v}_h,\mathbf{w}_h)-(\mathbf{f},\mathbf{w}_h)_Q\\
&= b_h(\mathbf{v}_h,\mathbf{w}_h)-b(\mathbf{u},\mathbf{w}_h)\\
&= (L_h\mathbf{v}_h,\mathbf{w}_h)_Q-(L\mathbf{u},\mathbf{w}_h)_Q\\
&= \big(L_h(\mathbf{v}_h-\mathbf{u}),\mathbf{w}_h\big)_Q-\big(L\mathbf{u},\mathbf{w}_h\big)_Q+\big(L_h\mathbf{u},\mathbf{w}_h\big)_Q\\
&= \big(L_h(\mathbf{v}_h-\mathbf{u}),\mathbf{w}_h\big)_Q-\big((M-M_h)\partial_t\mathbf{u},\mathbf{w}_h\big)_Q\\
&\qquad -\big((D-D_h)\mathbf{u},\mathbf{w}_h\big)_Q-\big((A-A_h)\mathbf{u},\mathbf{w}_h\big)_Q\\
&= \big(M_h\Pi_hM_h^{-1}L_h(\mathbf{v}_h-\mathbf{u}),\mathbf{w}_h\big)_Q\\
&\qquad -\big(M_hM_h^{-1}(M-M_h)\partial_t\mathbf{u},\mathbf{w}_h\big)_Q-\big(M_hM_h^{-1}(D-D_h)\mathbf{u},\mathbf{w}_h\big)_Q\\
&\le \Big(\big\|\Pi_hM_h^{-1}L_h(\mathbf{v}_h-\mathbf{u})\big\|_{W_h}+\big\|M_h^{-1}(M-M_h)\partial_t\mathbf{u}\|_{W_h}+\big\|M_h^{-1}(D-D_h)\mathbf{u}\|_{W_h}\Big)\|\mathbf{w}_h\|_{W_h}
\end{aligned}
$$

and thus the assertion follows from

$$
\begin{aligned}
\|\mathbf{u}-\mathbf{u}_h\|_{V_h} &\le \|\mathbf{u}-\mathbf{v}_h\|_{V_h}+\|\mathbf{v}_h-\mathbf{u}_h\|_{V_h}\\
&\le \|\mathbf{u}-\mathbf{v}_h\|_{V_h}+\beta^{-1}\sup_{\mathbf{w}_h\in W_h\setminus\{\mathbf{0}\}}\frac{b_h(\mathbf{v}_h-\mathbf{u}_h,\mathbf{w}_h)}{\|\mathbf{w}_h\|_{W_h}}\\
&\le \|\mathbf{u}-\mathbf{v}_h\|_{V_h}\\
&\qquad +\beta^{-1}\Big(\big\|\Pi_hM_h^{-1}L_h(\mathbf{v}_h-\mathbf{u})\big\|_{W_h}+\big\|M_h^{-1}(M-M_h)\partial_t\mathbf{u}\big\|_{W_h}+\big\|M_h^{-1}(D-D_h)\mathbf{u}\big\|_{W_h}\Big)\\
&\le (1+\beta^{-1})\|\mathbf{u}-\mathbf{v}_h\|_{V_h}+\beta^{-1}\big\|M_h^{-1/2}(M-M_h)\partial_t\mathbf{u}\big\|_Q+\beta^{-1}\big\|M_h^{-1/2}(D-D_h)\mathbf{u}\big\|_Q
\end{aligned}
$$

by the interpolation estimate (55) and

$$
\big\|M_h^{-1/2}(M-M_h)\partial_t\mathbf{u}\big\|_Q=\big\|M_h^{-1/2}(M-M_h)M^{-1/2}M^{1/2}\partial_t\mathbf{u}\big\|_Q\le\big\|M_h^{-1/2}(M-M_h)M^{-1/2}\big\|_\infty\|\partial_t\mathbf{u}\|_W\,. \qquad \square
$$

**Remark 31.** The estimate is derived for homogeneous initial and boundary conditions. It transfers to the inhomogeneous case if initial and boundary data $\mathbf{u}_0$ and $g_j$ can be extended to $\mathrm{H}(L,Q)$, i.e., if $\hat{\mathbf{u}}\in\mathrm{H}(L,Q)$ exists such that $\hat{\mathbf{u}}(0,x)=\mathbf{u}_0(x)$ for $x\in\Omega$ and $(\underline{A}_{\mathbf{n}}\hat{\mathbf{u}}(t,x))_j=g_j(t,x)$ and for $(t,x)\in(0,T)\times\Gamma_k$, $k=1,\dots,m$. Then, the approximation $\mathbf{u}_h\in V_h(\mathbf{u}_0)$ in the affine space (47) is computed by $b_h(\mathbf{u}_h,\mathbf{w}_h)=\langle\ell_h,\mathbf{w}_h\rangle$ for $\mathbf{w}_h\in W_h$, and the strong solution with inhomogeneous initial and boundary data is given by $\mathbf{u}=\tilde{\mathbf{u}}+\hat{\mathbf{u}}\in\mathrm{H}(L,Q)$, where $\tilde{\mathbf{u}}\in V$ solves $L\tilde{\mathbf{u}}=\mathbf{f}-L\hat{\mathbf{u}}$. Then, the result in Thm. 30 can be extended.

**Remark 32.** Since the norm (48) in $V_h+(\mathrm{H}^1(Q;\mathbb{R}^m)\cap V)$ is discrete in the derivatives, the topology in the space $V$ with respect to this norm is equivalent to the topology in $\mathrm{L}_2$ with mesh dependent bounds for the norm equivalence. Norm equivalence with respect to $\|\cdot\|_V$ is obtained in the limit: Let $(V_h)_{h\in\mathcal{H}}$ be a shape regular family of discrete spaces with $0\in\overline{\mathcal{H}}$ such that $(V_h\cap V)_{h\in\mathcal{H}}$ is dense in $V$. Then, defining $\|\mathbf{v}\|_{V_\mathcal{H}}=\sup_{h\in\mathcal{H}}\|\mathbf{v}\|_{V_h}$ yields a norm, and for sufficiently smooth functions $\mathbf{v}\in\mathrm{H}^1(Q;\mathbb{R}^m)\cap V$ this norm is equivalent to $\|\cdot\|_V$.

## 4.5. Convergence for weak solutions

Qualitative convergence estimates with respect to the norm in $V\subset\mathrm{H}(L,Q)$ require additional regularity, so that these estimates do not apply to weak solutions with discontinuities or singularities. For weak solutions without additional regularity we only can derive asymptotic convergence. Here, this is shown for simplicity only for homogeneous boundary data. The given data are the right-hand side $\mathbf{f}\in\mathrm{L}_2(Q;\mathbb{R}^m)$ and the initial value $\mathbf{u}_0\in Z$. We assume that Lem. 15 and dual consistency for $A_h$ in Lem. 25 is satisfied.
In the first step, we show that the inf-sup stability of the Petrov–Galerkin method yields a uniform a priori bound for the approximation. We define the approximation of the initial value $\mathbf{u}_{0,h}$ by $\mathbf{u}_{0,h}(t)=(1-t/t_1)\Pi_{n,h}\mathbf{u}_0$ for $t\in(0,t_1)$ in the first time interval and $\mathbf{u}_{0,h}(t)=\mathbf{0}$ for $t>t_1$, so that $V_h(\mathbf{u}_0)=\mathbf{u}_{0,h}+V_h$.

**Lemma 33.** *The discrete solution $\mathbf{u}_h\in V_h(\mathbf{u}_0)$ of the variational space-time equation*

$$
b_h(\mathbf{u}_h,\mathbf{w}_h)=\big(\mathbf{f},\mathbf{w}_h\big)_Q\,,\qquad \mathbf{w}_h\in W_h
$$

*is bounded by*

$$
\|\mathbf{u}_h\|_{W_h}\le 2T\,\|M_h^{-1}\Pi_h\mathbf{f}\|_{W_h}+(1+2T)\,\|\mathbf{u}_{0,h}\|_{V_h}\,.
$$

*Proof.* For $\mathbf{v}_h = \mathbf{u}_h - \mathbf{u}_{0,h} \in V_h$ the estimate $\|\mathbf{v}_h\|_{W_h} \le 2T\|\Pi_h M_h^{-1} L_h \mathbf{v}_h\|_{W_h}$ in Lem. 29 together with

$$\begin{aligned}\big(\Pi_h M_h^{-1} L_h \mathbf{v}_h, \mathbf{w}_h\big)_{W_h} &= \big(M_h^{-1} L_h \mathbf{v}_h, \mathbf{w}_h\big)_{W_h} = \big(L_h \mathbf{v}_h, \mathbf{w}_h\big)_Q = b_h(\mathbf{u}_h, \mathbf{w}_h) - b_h(\mathbf{u}_{0,h}, \mathbf{w}_h)\\ &= \big(\mathbf{f}, \mathbf{w}_h\big)_Q - b_h(\mathbf{u}_{0,h}, \mathbf{w}_h) = \big(\Pi_h \mathbf{f}, \mathbf{w}_h\big)_Q - b_h(\mathbf{u}_{0,h}, \mathbf{w}_h)\end{aligned}$$

for $\mathbf{w}_h \in W_h$ yields by duality

$$\begin{aligned}\|\mathbf{v}_h\|_{W_h} &\le 2T\|\Pi_h M_h^{-1} L_h \mathbf{v}_h\|_{W_h} = 2T \sup_{\mathbf{w}_h \in W_h \setminus \{\mathbf{0}\}} \frac{\big(\Pi_h M_h^{-1} L_h \mathbf{v}_h, \mathbf{w}_h\big)_{W_h}}{\|\mathbf{w}_h\|_{W_h}}\\ &= 2T \sup_{\mathbf{w}_h \in W_h \setminus \{\mathbf{0}\}} \frac{\big(\Pi_h \mathbf{f}, \mathbf{w}_h\big)_Q - b_h(\mathbf{u}_{0,h}, \mathbf{w}_h)}{\|\mathbf{w}_h\|_{W_h}} \le 2T \left(\|M_h^{-1}\Pi_h \mathbf{f}\|_{W_h} + \|\Pi_h M_h^{-1} L_h \mathbf{u}_{0,h}\|_{W_h}\right),\end{aligned}$$

so that

$$\begin{aligned}\|\mathbf{u}_h\|_{W_h} &\le \|\mathbf{v}_h\|_{W_h} + \|\mathbf{u}_{0,h}\|_{W_h} \le 2T \left(\|M_h^{-1}\Pi_h \mathbf{f}\|_{W_h} + \|\Pi_h M_h^{-1} L_h \mathbf{u}_{0,h}\|_{W_h}\right) + \|\mathbf{u}_{0,h}\|_{W_h}\\ &\le 2T \,\|M_h^{-1}\Pi_h \mathbf{f}\|_{W_h} + (1 + 2T)\, \|\mathbf{u}_{0,h}\|_{V_h}\,.\end{aligned}$$

□

Next we show that the dual consistency of the DG operator implies dual consistency of the space-time method. For simplicity, we assume that the parameters in $M$ and $D$ are piecewise constant on all cells $K \in \mathcal{K}_h$, so that $M = M_h$ and $D = D_h$, which implies $\|\mathbf{z}_h\|_{W_h} = \|\mathbf{z}_h\|_W$ for $\mathbf{z}_h \in \mathrm{L}_2(0,T;Y_h)$.

**Lemma 34.** *We have*

$$b_h(\mathbf{v}_h, \mathbf{w}) = \big(\mathbf{v}_h, L^*\mathbf{w}\big)_Q - \big(M\Pi_{1,h}\mathbf{u}_0, \mathbf{w}\big)_\Omega\,,\ \ \mathbf{v}_h \in V_h(\mathbf{u}_0)\,,\ \ \mathbf{w} \in W_h \cap V^*\,. \tag{56}$$

*Proof.* We obtain for $\mathbf{v}_h \in V_h(\mathbf{u}_0) \subset \mathrm{H}^1(0,T;Y_h)$ and $\mathbf{w} \in W_h \cap V^*$

$$\begin{aligned}b_h(\mathbf{v}_h, \mathbf{w}) &= \big(M\partial_t \mathbf{v}_h, \mathbf{w}\big)_Q + \big(D\mathbf{v}_h, \mathbf{w}\big)_Q + \big(A_h \mathbf{v}_h, \mathbf{w}\big)_{Q_h}\\ &= \big(M\mathbf{v}_h(T), \mathbf{w}(T)\big)_\Omega - \big(M\mathbf{v}_h(0), \mathbf{w}(0)\big)_\Omega - \big(\mathbf{v}_h, M\partial_t \mathbf{w}\big)_Q + \big(\mathbf{v}_h, D\mathbf{w}\big)_Q - \big(\mathbf{v}_h, A\mathbf{w}\big)_Q\\ &= \big(\mathbf{v}_h, L^*\mathbf{w}\big)_Q - \big(M\Pi_{1,h}\mathbf{u}_0, \mathbf{w}(0)\big)_\Omega\,.\end{aligned}$$

using $M = M_h$, $D = D_h$, integration by parts for $\mathbf{v}_h, \mathbf{w} \in \mathrm{H}^1(0,T;Y_h)$, $\mathbf{w}(T) = \mathbf{0}$ in $V^*$, and the dual consistency of the DG operator $A_h$ with upwind flux (see Lem. 24 and Lem. 25 for acoustics). □

Let $(V_h, W_h)$, $h \in \mathcal{H} \subset (0, h_0)$, be a dense family of nested discretizations with $V_h \subset V_{h'}$ and $W_h \subset W_{h'}$ for $h' < h$, $h, h' \in \mathcal{H}$ and $0 \in \overline{\mathcal{H}}$. We assume that the assumptions in this section are fulfilled for all discretizations, so that $(V_h, W_h)$ is uniformly inf-sup stable by Thm. 26. We only consider the case $\mathbb{P}_1(Q_h;\mathbb{R}^m) \subset W_h$, so that $W_h \cap \mathrm{H}^1(Q;\mathbb{R}^m)$ includes the continuous linear elements and thus $\bigcup_{h\in\mathcal{H}}(V^* \cap W_h)$ is dense in $V^*$.

**Theorem 35.** *Assume $M = M_h$, $D = D_h$, and that $\|\mathbf{u}_{0,h}\|_{V_h} \le C$ is uniformly bounded for all $h \in \mathcal{H}$. Then, the discrete solutions $(\mathbf{u}_h)_{h\in\mathcal{H}}$ are weakly converging to the weak solution $\mathbf{u} \in W$ of the equation*

$$(\mathbf{u}, L^*\mathbf{z})_Q = \big(\mathbf{f}, \mathbf{z}\big)_Q + \big(M\mathbf{u}_0, \mathbf{z}(0)\big)_\Omega\,, \qquad \mathbf{z} \in \mathcal{V}^*\,. \tag{57}$$

*Proof.* By Thm. 26 $\mathbf{u}_h - \mathbf{u}_{0,h}$ is uniformly bounded in $V_h$ and thus, by Lem. 29, $(\mathbf{u}_h)_{h\in\mathcal{H}}$ is uniformly bounded in $W$, so that a subsequence $\mathcal{H}_0 \subset \mathcal{H}$ and a weak limit $\mathbf{u} \in W$ exists, i.e.,

$$\lim_{h\in\mathcal{H}_0} \big(\mathbf{u}_h, \mathbf{w}\big)_W = \big(\mathbf{u}, \mathbf{w}\big)_W\,, \qquad \mathbf{w} \in W\,.$$

The assumption that $(W_h \cap V^*)_{h\in\mathcal{H}}$ is dense in $V^* \supset \mathcal{V}^*$ implies that for all $\mathbf{z} \in \mathcal{V}^*$ there exists a sequence $(\mathbf{w}_h)_{h\in\mathcal{H}}$ with $\mathbf{w}_h \in W_h \cap V^*$ and $\lim\limits_{h\in\mathcal{H}} \|\mathbf{w}_h - \mathbf{z}\|_{V^*} = 0$. This implies $\lim\limits_{h\in\mathcal{H}} \|\mathbf{w}_h(0) - \mathbf{z}(0)\|_Y = 0$, cf. Rem. 16. Using the weak convergence of $\mathbf{u}_h$, the strong convergence of $L^*\mathbf{w}_h$, and Lem. 34 yields

$$\begin{aligned}\big(\mathbf{u}, L^*\mathbf{z}\big)_Q &= \lim_{h\in\mathcal{H}_0} \big(\mathbf{u}_h, L^*\mathbf{z}\big)_Q = \lim_{h\in\mathcal{H}_0} \big(\mathbf{u}_h, L^*\mathbf{w}_h\big)_Q = \lim_{h\in\mathcal{H}_0} \Big(b_h(\mathbf{u}_h, \mathbf{w}_h) + \big(M\Pi_{1,h}\mathbf{u}_0, \mathbf{w}_h(0)\big)_\Omega\Big)\\ &= \lim_{h\in\mathcal{H}_0} \big(\mathbf{f}, \mathbf{w}_h\big)_Q + \big(M\mathbf{u}_0, \mathbf{z}(0)\big)_\Omega = \big(\mathbf{f}, \mathbf{z}\big)_Q + \big(M\mathbf{u}_0, \mathbf{z}(0)\big)_\Omega\,,\end{aligned}$$

so that $\mathbf{u}$ is a weak solution of (57). Since the weak solution is unique by Thm. 11 and Lem. 15, this shows that the weak limit of all subsequences in $\mathcal{H}$ is the unique weak solution, so that the full sequence is convergent. $\square$

**Remark 36.** Since we assume that $(\mathbf{u}_{0,h})_{h\in\mathcal{H}}$ is uniformly bounded in $V_h$, the initial value $\mathbf{u}_0$ extends to $\mathrm{H}(L,Q)$, and the weak solution is a strong solution.

## 4.6. Goal-oriented adaptivity

In order to find an efficient choice for the polynomial degrees $(p_R, q_R)$, we introduce a dual-weighted residual error indicator with respect to a suitable goal functional. Its construction is based on a dual-primal error representation combined with a priori estimates constructed from an approximation of the dual solution. Note that this corresponds to a problem backward in time, so that the resulting error indicator only refines regions of the space-time domain which are relevant for the evaluation of the chosen goal functional.

**Dual-primal error bound.** Let $E\colon W \longrightarrow \mathbb{R}$ be a linear error functional. Our goal is to estimate and then to reduce the error with respect to this functional. The dual solution $\mathbf{u}^* \in V^*$ is defined by

$$(\mathbf{w}, L^*\mathbf{u}^*)_Q = \langle E, \mathbf{w}\rangle\,, \qquad \mathbf{w} \in W\,.$$

For the local representation of $E$ we define the pairing in $R \in \mathcal{R}_h$

$$\langle \mathbf{v}_R, \mathbf{w}_R\rangle_{\partial R} = (L\mathbf{v}_R, \mathbf{w}_R)_R - (\mathbf{v}_R, L^*\mathbf{w}_R)_R\,, \qquad \mathbf{v}_R \in \mathrm{H}(L,R)\,,\ \mathbf{w}_R \in \mathrm{H}(L^*,R)\,.$$

**Lemma 37.** *Let $\mathbf{u} \in V$ be the solution of $L\mathbf{u} = \mathbf{f}$, and let $\mathbf{u}_h \in V_h$ be the approximation solving (49). Then, the error can be represented by*

$$\langle E, \mathbf{u} - \mathbf{u}_h\rangle = \sum_{R\in\mathcal{R}_h} \Big( \big(\mathbf{f} - L\mathbf{u}_h, \mathbf{u}^*\big)_R + \langle \mathbf{u}_h, \mathbf{u}^*\rangle_{\partial R}\Big)\,.$$

*If the dual solution is sufficiently regular, the error is bounded for all $\mathbf{w}_h \in W_h$ by*

$$\begin{aligned}
\big|\langle E, \mathbf{u} - \mathbf{u}_h\rangle\big| \le \sum_{n=1}^{N} \sum_{R=(t_{n-1},t_n)\times K} &\Bigg( \big\|\mathbf{f} - (M_h\partial_t + A + D_h)\mathbf{u}_h\big\|_R \big\|\mathbf{u}^* - \mathbf{w}_h\big\|_R \\
&\quad + \sum_{F\in\mathcal{F}_K} \big\|\underline{A}_{\mathbf{n}_K}\mathbf{u}_{h,R} - \underline{A}^{\mathrm{upw}}_{\mathbf{n}_K}\mathbf{u}_{n,h}\big\|_{(t_{n-1},t_n)\times F} \big\|\mathbf{u}^* - \mathbf{w}_h\big\|_{(t_{n-1},t_n)\times F}\Bigg) \\
&+ \sum_{n=1}^{N-1} \big\|M_h\big(\mathbf{u}_{n,h}(t_n) - \Pi_{n+1,h}\mathbf{u}_{n,h}(t_n)\big)\big\|_\Omega \big\|\mathbf{u}^*(t_n) - \mathbf{w}_{n+1,h}(t_n)\big\|_\Omega \\
&+ \big\|M_h^{-1/2}(M - M_h)M^{-1/2}\|_\infty \sum_{n=1}^{N-1} \|\mathbf{u}_{n,h}(t_n) - \Pi_{n+1,h}\mathbf{u}_{n,h}(t_n)\|_Y \|\mathbf{u}^*(t_n)\|_Y \\
&+ \big\|M_h^{-1/2}(M - M_h)M^{-1/2}\|_\infty \big\|\partial_t\mathbf{u}_h\big\|_W \big\|\mathbf{u}^*\big\|_W \\
&+ \big\|M_h^{-1/2}(D - D_h)M^{-1/2}\|_\infty \big\|\mathbf{u}_h\big\|_W \big\|\mathbf{u}^*\big\|_W\,.
\end{aligned}$$

*Proof.* We have by definition of $\mathbf{u}^*$

$$\begin{aligned}
\langle E, \mathbf{u} - \mathbf{u}_h\rangle &= (\mathbf{u} - \mathbf{u}_h, L^*\mathbf{u}^*)_Q = (\mathbf{u}, L^*\mathbf{u}^*)_Q - (\mathbf{u}_h, L^*\mathbf{u}^*)_Q \\
&= (L\mathbf{u}, \mathbf{u}^*)_Q - (\mathbf{u}_h, L^*\mathbf{u}^*)_Q = (\mathbf{f}, \mathbf{u}^*)_Q - \sum_{R\in\mathcal{R}_h} (\mathbf{u}_h, L^*\mathbf{u}^*)_R \\
&= (\mathbf{f}, \mathbf{u}^*)_Q - \sum_{R\in\mathcal{R}_h} \Big( (L\mathbf{u}_h, \mathbf{u}^*)_R - \langle \mathbf{u}_h, \mathbf{u}^*\rangle_{\partial R}\Big) \\
&= \sum_{R\in\mathcal{R}_h} \Big( \big(\mathbf{f} - L\mathbf{u}_h, \mathbf{u}^*\big)_R + \langle \mathbf{u}_h, \mathbf{u}^*\rangle_{\partial R}\Big)\,.
\end{aligned}$$

Using $\mathbf{u}_h(0) = \mathbf{0}$ and $\mathbf{u}^*(T) = \mathbf{0}$, we obtain

$$\begin{aligned}
\sum_{n=1}^{N} \sum_{R=(t_{n-1},t_n)\times K} (M\partial_t \mathbf{u}_{n,h}, \mathbf{u}^*)_R + (M\mathbf{u}_{n,h}, \partial_t \mathbf{u}^*)_R &= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \partial_t (M\mathbf{u}_{n,h}, \mathbf{u}^*)_\Omega \,\mathrm{d}t \\
&= \sum_{n=1}^{N} \big(M\mathbf{u}_{n,h}(t_n), \mathbf{u}^*(t_n)\big)_\Omega - \big(M\mathbf{u}_{n,h}(t_{n-1}), \mathbf{u}^*(t_{n-1})\big)_\Omega \\
&= \sum_{n=1}^{N-1} \Big(M\big(\mathbf{u}_{n,h}(t_n) - \mathbf{u}_{n+1,h}(t_n)\big), \mathbf{u}^*(t_n)\Big)_\Omega \\
&= \sum_{n=1}^{N-1} \Big(M\big(\mathbf{u}_{n,h}(t_n) - \Pi_{n+1,h}\mathbf{u}_{n,h}(t_n)\big), \mathbf{u}^*(t_n)\Big)_\Omega
\end{aligned}$$

and in every time slice $(t_{n-1}, t_n)$ we obtain, if the dual solution $\mathbf{u}^*$ is sufficiently smooth satisfying $\mathbf{u}^*|_{\partial Q_h} \in \mathrm{L}_2(\partial Q_h; \mathbb{R}^m)$, for the restriction to the space-time skeleton

$$\begin{aligned}
\sum_{K\in\mathcal{K}_h} \big(A\mathbf{u}_{n,h}, \mathbf{u}^*\big)_{(t_{n-1},t_n)\times K} &+ \big(\mathbf{u}_{n,h}, A\mathbf{u}^*\big)_{(t_{n-1},t_n)\times K} \\
&= \sum_{K\in\mathcal{K}_h} \big(\underline{A}_{\mathbf{n}_K} \mathbf{u}_{n,h.K}, \mathbf{u}^*\big)_{(t_{n-1},t_n)\times \partial K} \\
&= \sum_{K\in\mathcal{K}_h} \sum_{F\in\mathcal{F}_K} \big(\underline{A}_{\mathbf{n}_K} \mathbf{u}_{n,h.K}, \mathbf{u}^*\big)_{(t_{n-1},t_n)\times F} \\
&= \sum_{K\in\mathcal{K}_h} \sum_{F\in\mathcal{F}_K} \big(\underline{A}_{\mathbf{n}_K} \mathbf{u}_{n,h,K} - \underline{A}_{\mathbf{n}_K}^{\mathrm{upw}} \mathbf{u}_{n,h}, \mathbf{u}^*\big)_{(t_{n-1},t_n)\times F}
\end{aligned}$$

where $\mathbf{u}_{n,h,K}$ is the extension of $\mathbf{u}_{n,h}|_K$ to $\overline{K}$. This gives, inserting (37),

$$\begin{aligned}
\sum_{R\in\mathcal{R}_h} \langle \mathbf{u}_h, \mathbf{u}^*\rangle_{\partial R} &= \sum_{n=1}^{N} \sum_{R=(t_{n-1},t_n)\times K} \big(L\mathbf{u}_{n,h}, \mathbf{u}^*\big)_R - \big(\mathbf{u}_{n,h}, L^*\mathbf{u}^*\big)_R \\
&= \sum_{n=1}^{N} \sum_{R=(t_{n-1},t_n)\times K} \big(M\partial_t \mathbf{u}_{n,h}, \mathbf{u}^*\big)_R + \big(M\mathbf{u}_{n,h}, \partial_t\mathbf{u}^*\big)_R + \big(A\mathbf{u}_{n,h}, \mathbf{u}^*\big)_R + \big(\mathbf{u}_{n,h}, A\mathbf{u}^*\big)_R \\
&= \sum_{n=1}^{N-1} \Big(M\big(\mathbf{u}_{n,h}(t_n) - \Pi_{n+1,h}\mathbf{u}_{n,h}(t_n)\big), \mathbf{u}^*(t_n)\Big)_\Omega \\
&\qquad + \sum_{R=(t_{n-1},t_n)\times K} \sum_{F\in\mathcal{F}_K} \big(\underline{A}_{\mathbf{n}_K} \mathbf{u}_{h,R} - \underline{A}_{\mathbf{n}_K}^{\mathrm{upw}} \mathbf{u}_{n,h}, \mathbf{u}^*\big)_{(t_{n-1},t_n)\times F}\,,
\end{aligned}$$

where $\mathbf{u}_{h,R}$ is the extension of $\mathbf{u}_h|_R$ to $\overline{R}$.
For the discrete solution $\mathbf{u}_h \in V_h$ and any discrete test function $\mathbf{w}_h \in W_h$ we have

$$\begin{aligned}
\big(\mathbf{f}, \mathbf{w}_h\big)_Q &= \big(L_h\mathbf{u}_h, \mathbf{w}_h\big)_Q = \big(M_h\partial_t\mathbf{u}_h, \mathbf{w}_h\big)_Q + \big(A_h\mathbf{u}_h, \mathbf{w}_h\big)_Q + \big(D_h\mathbf{u}_h, \mathbf{w}_h\big)_Q \\
&= \big(M_h\partial_t\mathbf{u}_h, \mathbf{w}_h\big)_Q + \big(D_h\mathbf{u}_h, \mathbf{w}_h\big)_Q \\
&\quad + \sum_{n=1}^{N} \sum_{R=(t_{n-1},t_n)\times K} \Bigg( \big(A\mathbf{u}_h, \mathbf{w}_h\big)_R + \sum_{F\in\mathcal{F}_K} \big(\underline{A}_{\mathbf{n}_K}^{\mathrm{upw}} \mathbf{u}_{n,h} - \underline{A}_{\mathbf{n}_K} \mathbf{u}_{h,R}, \mathbf{w}_{h,R}\big)_{(t_{n-1},t_n)\times F} \Bigg),
\end{aligned}$$

so that

$$\begin{aligned}
0 = \sum_{n=1}^{N} \sum_{R=(t_{n-1},t_n)\times K} \Bigg( &\big(A\mathbf{u}_h - \mathbf{f}, \mathbf{w}_h\big)_R + \big(M_h\partial_t\mathbf{u}_h, \mathbf{w}_h\big)_R + \big(D_h\mathbf{u}_h, \mathbf{w}_h\big)_R \\
&+ \sum_{F\in\mathcal{F}_K} \big(\underline{A}_{\mathbf{n}_K}^{\mathrm{upw}} \mathbf{u}_{n,h} - \underline{A}_{\mathbf{n}_K} \mathbf{u}_{h,R}, \mathbf{w}_{h,R}\big)_{(t_{n-1},t_n)\times F} \Bigg).
\end{aligned}$$

Together, this gives

$$
\begin{aligned}
\langle E, \mathbf{u} - \mathbf{u}_h \rangle &= \sum_{R\in\mathcal{R}_h} \Big( \big(\mathbf{f} - L\mathbf{u}_h, \mathbf{u}^*\big)_R + \langle \mathbf{u}_h, \mathbf{u}^* \rangle_{\partial R} \Big) \\
&= \sum_{n=1}^{N} \sum_{R=(t_{n-1},t_n)\times K} \Bigg( \big(\mathbf{f} - (M_h\partial_t + A + D_h)\mathbf{u}_h, \mathbf{u}^*\big)_R \\
&\qquad\qquad - \big((M - M_h)\partial_t \mathbf{u}_h, \mathbf{u}^*\big)_R - \big((D - D_h)\mathbf{u}_h, \mathbf{u}^*\big)_R \\
&\qquad\qquad + \sum_{F\in\mathcal{F}_K} \big(\underline{A}_{\mathbf{n}_K}\mathbf{u}_{h,R} - \underline{A}_{\mathbf{n}_K}^{\mathrm{upw}}\mathbf{u}_{n,h}, \mathbf{u}^*\big)_{(t_{n-1},t_n)\times F} \Bigg) \\
&\quad + \sum_{n=1}^{N-1} \big( M\big(\mathbf{u}_{n,h}(t_n) - \Pi_{n+1,h}\mathbf{u}_{n,h}(t_n)\big), \mathbf{u}^*(t_n)\big)_\Omega \\
&= \sum_{n=1}^{N} \sum_{R=(t_{n-1},t_n)\times K} \Bigg( \big(\mathbf{f} - (M_h\partial_t + A + D_h)\mathbf{u}_h, \mathbf{u}^* - \mathbf{w}_h\big)_R \\
&\qquad\qquad + \sum_{F\in\mathcal{F}_K} \big(\underline{A}_{\mathbf{n}_K}\mathbf{u}_{h,R} - \underline{A}_{\mathbf{n}_K}^{\mathrm{upw}}\mathbf{u}_{n,h}, \mathbf{u}^* - \mathbf{w}_h\big)_{(t_{n-1},t_n)\times F} \Bigg) \\
&\quad + \sum_{n=1}^{N-1} \big( M_h\big(\mathbf{u}_{n,h}(t_n) - \Pi_{n+1,h}\mathbf{u}_{n,h}(t_n)\big), \mathbf{u}^*(t_n) - \mathbf{w}_{n+1,h}(t_n)\big)_\Omega \\
&\quad + \sum_{n=1}^{N-1} \big( (M - M_h)\big(\mathbf{u}_{n,h}(t_n) - \Pi_{n+1,h}\mathbf{u}_{n,h}(t_n)\big), \mathbf{u}^*(t_n)\big)_\Omega \\
&\quad - \big((M - M_h)\partial_t\mathbf{u}_h, \mathbf{u}^*\big)_Q - \big((D - D_h)\mathbf{u}_h, \mathbf{u}^*\big)_Q \\
&\le \sum_{n=1}^{N} \sum_{R=(t_{n-1},t_n)\times K} \Bigg( \big\|\mathbf{f} - (M_h\partial_t + A + D_h)\mathbf{u}_h\big\|_R \big\|\mathbf{u}^* - \mathbf{w}_h\big\|_R \\
&\qquad\qquad + \sum_{F\in\mathcal{F}_K} \big\|\underline{A}_{\mathbf{n}_K}\mathbf{u}_{h,R} - \underline{A}_{\mathbf{n}_K}^{\mathrm{upw}}\mathbf{u}_{n,h}\big\|_{(t_{n-1},t_n)\times F} \big\|\mathbf{u}^* - \mathbf{w}_h\big\|_{(t_{n-1},t_n)\times F} \Bigg) \\
&\quad + \sum_{n=1}^{N-1} \big\|M_h\big(\mathbf{u}_{n,h}(t_n) - \Pi_{n+1,h}\mathbf{u}_{n,h}(t_n)\big)\big\|_\Omega \big\|\mathbf{u}^*(t_n) - \mathbf{w}_{n+1,h}(t_n)\big\|_\Omega \\
&\quad + \sum_{n=1}^{N-1} \big\|M^{-1}(M - M_h)\big(\mathbf{u}_{n,h}(t_n) - \Pi_{n+1,h}\mathbf{u}_{n,h}(t_n)\big)\|_Y \|\mathbf{u}^*(t_n)\|_Y \\
&\quad + \big\|(M - M_h)\partial_t\mathbf{u}_h\big\|_{W^*} \big\|\mathbf{u}^*\big\|_W - \big\|(D - D_h)\mathbf{u}_h\big\|_{W^*} \big\|\mathbf{u}^*\big\|_W \,.
\end{aligned}
$$

This yields the assertion. □

**Dual-primal error indicator.** For the evaluation of the error bound the exact solution $\mathbf{u}^*$ of the dual problem is required, and for $\mathbf{w}_h$ an interpolation of $\mathbf{u}^*$ can be inserted. Then, the interpolation errors $\big\|\mathbf{u}^* - \mathbf{w}_h\big\|_R$ and $\big\|\mathbf{u}^* - \mathbf{w}_h\big\|_{(t_{n-1},t_n)\times F}$ can be estimated by the regularity of the dual solution.
Since $\mathbf{u}^* \in V^*$ cannot be computed exactly, it is approximated by $\mathbf{u}_h^* \in W_h$ solving the discrete dual solution

$$b_h(\mathbf{v}_h, \mathbf{u}_h^*) = \langle E, \mathbf{v}_h \rangle\,, \qquad \mathbf{v}_h \in V_h\,,$$

and the regularity of the dual solution is estimated from the regularity of $\mathbf{u}_h^*$. Therefore, we compute the $\mathrm{L}_2$ projection $\Pi_h^0\colon \mathrm{L}_2(Q;\mathbb{R}^m) \longrightarrow \mathbb{P}_0(Q_h;\mathbb{R}^m)$ and the jump terms $[\Pi_h^0\mathbf{u}_h^*]_F$ with $[\mathbf{y}_h]_F = \mathbf{y}_{h,K_F} - \mathbf{y}_{h,K}$ on inner faces, $([\mathbf{y}_h]_F)_j = (\underline{A}_{\mathbf{n}}\mathbf{y}_h)_j$ for $F \subset \Gamma_j^*$, and $([\mathbf{y}_h]_F)_j = \mathbf{0}$ for $F \subset \partial\Omega \setminus \Gamma_j^*$. Then, the error indicator $\eta_h = \sum_{R\in\mathcal{R}_h} \eta_R$ for $R = (t_{n-1}, t_n) \times K$ is defined by

$$\begin{aligned}\eta_R = \Big(\big\|(M_h\partial_t + A + D_h)\mathbf{u}_h - \mathbf{f}\big\|_R + \big\|\mathbf{u}_{n,h}(t_{n-1}) - \Pi_{n,h}\mathbf{u}_{n-1,h}(t_{n-1})\|_K\Big)\; h_K^{1/2}\;\big\|[\Pi_h^0\mathbf{u}_h^*]_F\big\|_{(t_{n-1},t_n)\times\partial K}\\ + \big\|\big(\underline{A}_{\mathbf{n}_K} - \underline{A}_{\mathbf{n}_K}^{\mathrm{upw}}\big)\mathbf{u}_h\|_{(t_{n-1},t_n)\times\partial K}\big\|[\Pi_h^0\mathbf{u}_h^*]_F\big\|_{(t_{n-1},t_n)\times\partial K}\,.\end{aligned}$$

Depending on threshold parameters $0 < \vartheta_0 < \vartheta_0 < 1$ this results in the following $p$-adaptive algorithm:

1: choose low order polynomial degrees on the initial mesh
2: **while** $\max_R(p_R) \le p_{\max}$ and $\max_R(q_R) \le q_{\max}$ **do**
3: compute $\mathbf{u}_h$
4: compute $\mathbf{u}_h^*$ and the projection $\Pi_h^0\mathbf{u}_h^*$
5: compute $\eta_R$ on every cell $R$
6: if the estimated error $\eta_h$ is small enough, then STOP
7: mark space-time cell $R$ for refinement if $\eta_R > \vartheta_1 \max_{R'} \eta_{R'}$
and for derefinement if $\eta_R < \vartheta_0 \max_{R'} \eta_{R'}$
8: increase/decrease polynomial degrees on marked cells
9: redistribute cells on processes for better load balancing

### 4.7. **Reliable error estimation for weak solutions**

Finally, we derive a posteriori estimates for weak solutions based on local conforming reconstructions. Here we consider the general case including inhomogeneous initial and boundary data, where initial data are included in the definition of the affine ansatz space (47), and the DG formulation for boundary data is derived in (38), see (45) for an example. For simplicity, we assume that the parameters in $M$ and $D$ are piecewise constant, so that $M = M_h$ and $D = D_h$.
For the data $\mathbf{f} \in \mathrm{L}_2(Q;\mathbb{R}^m)$, $\mathbf{u}_0 \in \mathrm{L}_2(\Omega;\mathbb{R}^m)$, $g_k \in \mathrm{L}_2((0,T)\times\Gamma_k)$ defining the linear functional $\ell$ by

$$\langle \ell, \mathbf{z}\rangle = (\mathbf{f},\mathbf{z})_Q + \big(M\mathbf{u}_0, \mathbf{z}(0)\big)_\Omega - \sum_{k=1}^m \big(g_k, z_k\big)_{(0,T)\times\Gamma_k}\,, \qquad \mathbf{z}\in\mathcal{V}^*\,,$$

we select piecewise polynomial approximations $\mathbf{f}_h \in \mathbb{P}(Q_h;\mathbb{R}^m)$, $\mathbf{u}_{0,h}\in\mathbb{P}(\Omega_h;\mathbb{R}^m)$, and $g_{k,h}\in\mathbb{P}((0,T)\times\Gamma_k)$ defining the approximated linear functional $\ell_h$ by

$$\langle \ell_h, \mathbf{z}_h\rangle = (\mathbf{f}_h,\mathbf{z}_h)_Q + \big(M\mathbf{u}_{0,h}, \mathbf{z}_h(0)\big)_\Omega - \sum_{k=1}^m \big(g_{k,h}, z_{k,h}\big)_{(0,T)\times\Gamma_k}\,, \qquad \mathbf{z}_h\in\mathcal{V}^*\,.$$

We assume that $\ell$ is bounded by (29) so that a unique weak solution $\mathbf{u}\in W$ of

$$\big(\mathbf{u}, L^*\mathbf{z}\big)_Q = \langle\ell,\mathbf{z}\rangle\,, \qquad \mathbf{z}\in\mathcal{V}^*$$

exists by Thm. 11 and Cor. 20. For the approximation $\mathbf{u}_h\in V_h(\mathbf{u}_{0,h})$ solving

$$b_h(\mathbf{u}_h,\mathbf{w}_h) = \big(\mathbf{f}_h,\mathbf{w}_h\big)_Q - \big(\underline{A}_{\mathbf{n}}^{\mathrm{bnd}}\mathbf{g}_h, \mathbf{w}_h\big)_{(0,T)\times\partial\Omega}\,, \qquad \mathbf{w}_h\in W_h\,,$$

we now construct a conforming reconstruction in a continuous finite element space $V_h^{\mathrm{cf}}\subset \mathrm{H}(L,Q)\cap\mathbb{P}(Q_h;\mathbb{R}^m)$ as described in the following. Here, we set for the right-hand side $\mathbf{g}_h = (g_{k,h})_{k=1,\dots,m}\in\mathrm{L}_2((0,T)\times\partial\Omega;\mathbb{R}^m)$ with $g_{k,h}=0$ on $\partial\Omega\setminus\Gamma_k$.
The reconstruction is defined on local patches associated to the corners of the space-time mesh. Therefore, let $\mathcal{C}_K\subset\overline{K}$ be the corner points in space of the elements $K\in\mathcal{K}_h$ such that $\overline{K} = \operatorname{conv}\mathcal{C}_K$, and define $\mathcal{C}_h = \bigcup_{K\in\mathcal{K}_h}\mathcal{C}_K$. For all $\mathbf{c}\in\mathcal{C}_h$ we define $\mathcal{K}_{h,\mathbf{c}} = \big\{K\in\mathcal{K}_h\colon \mathbf{c}\in\mathcal{C}_K\big\}$ and open subdomains $\omega_{\mathbf{c}}\subset\Omega$ with $\overline{\omega}_{\mathbf{c}} = \bigcup_{K\in\mathcal{K}_{h,\mathbf{c}}}\overline{K}$. This extends to space-time patches $Q_{0,\mathbf{c}} = (0,t_1)\times\omega_{\mathbf{c}}$, $Q_{n,\mathbf{c}} = (t_{n-1},t_{n+1})\times\omega_{\mathbf{c}}$ for $n=1,\dots,N-1$, and $Q_{N,\mathbf{c}} = (t_{N-1},T)\times\omega_{\mathbf{c}}$. Let $\psi_{n,\mathbf{c}}\in\mathrm{C}^0(\overline{Q})\cap\mathbb{P}(Q_h)$ be a corresponding decomposition of $1\equiv\sum_{n=0}^N\sum_{\mathbf{c}\in\mathcal{C}_h}\psi_{n,\mathbf{c}}$ with $\operatorname{supp}\psi_{n,\mathbf{c}} = \overline{Q}_{n,\mathbf{c}}$. On every patch we define discrete conforming local affine spaces

$$\begin{aligned}
V_{n,\mathbf{c}}^{\mathrm{cf}}(\mathbf{u}_{0,h},\mathbf{g}_h) = \big\{\mathbf{v}_h\in V_h^{\mathrm{cf}}\colon\ & \operatorname{supp}(\mathbf{v}_h)\subset\overline{Q}_{n,\mathbf{c}}\,,\\
&\mathbf{v}_h(0) = \psi_{n,\mathbf{c}}\mathbf{u}_{0,h} \text{ in } \Omega \text{ if } n=0\,,\\
&\mathbf{v}_h(t_{n-1}) = \mathbf{0} \text{ in } \Omega \text{ if } n>0\,,\\
&\mathbf{v}_h(t_{n+1}) = \mathbf{0} \text{ in } \Omega \text{ if } n<N\,,\\
&(\underline{A}_{\mathbf{n}}\mathbf{v}_h)_k = \psi_{n,\mathbf{c}}g_{k,h} \text{ on } (0,T)\times\Gamma_k\,,\ k=1,\dots,m\,,\\
&\underline{A}_{\mathbf{n}}\mathbf{v}_h = \mathbf{0} \text{ on } (0,T)\times(\partial\omega_{\mathbf{c}}\setminus\partial\Omega)\big\}\,.
\end{aligned}$$

In the following we assume $V_{n,\mathbf{c}}^{\mathrm{cf}}(\mathbf{u}_{0,h},\mathbf{g}_h)\neq\emptyset$, which can be achieved by a suitable choice of the data approximation $\mathbf{u}_{0,h}$ and $\mathbf{g}_{k,h}$ depending on the reconstruction space $V_h^{\mathrm{cf}}$.
Now, the local conforming reconstruction of the discrete solution $\mathbf{u}_h$ is defined by $\mathbf{u}_h^{\mathrm{cf}} = \sum_{n=0}^N\sum_{\mathbf{c}\in\mathcal{C}_h}\mathbf{u}_{n,\mathbf{c}}^{\mathrm{cf}}$, where $\mathbf{u}_{n,\mathbf{c}}^{\mathrm{cf}}\in V_{n,\mathbf{c}}^{\mathrm{cf}}(\mathbf{u}_{0,h},\mathbf{g}_h)$ is the best approximation of $\psi_{n,\mathbf{c}}\mathbf{u}_h$ in the topology of $W$, i.e.,

$$\big\|\psi_{n,\mathbf{c}}\mathbf{u}_h - \mathbf{u}_{n,\mathbf{c}}^{\mathrm{cf}}\big\|_W \le \big\|\psi_{n,\mathbf{c}}\mathbf{u}_h - \mathbf{v}_{n,\mathbf{c}}\big\|_W\,, \qquad \mathbf{v}_{n,\mathbf{c}}\in V_{n,\mathbf{c}}^{\mathrm{cf}}(\mathbf{u}_{0,h},\mathbf{g}_h)\,,$$

so that $\mathbf{u}_{n,\mathbf{c}}^{\mathrm{cf}}$ is determined by a small local quadratic minimization problem with linear constraints.

**Lemma 38.** *The approximation error of the weak solution can be estimated by*

$$\big\|\mathbf{u}-\mathbf{u}_h\big\|_W \le \big\|\mathbf{u}_h-\mathbf{u}_h^{\mathrm{cf}}\big\|_W + 2T\,\big\|L\mathbf{u}_h^{\mathrm{cf}}-\mathbf{f}_h\big\|_{W^*} + \beta^{-1}\sup_{\mathbf{z}\in\mathcal{V}^*\setminus\{\mathbf{0}\}}\frac{\langle \ell-\ell_h,\mathbf{z}\rangle}{\|\mathbf{z}\|_{V^*}}\,.$$

*Proof.* By construction we have for $\mathbf{u}_{n,\mathbf{c}}^{\mathrm{cf}}\in V_{n,\mathbf{c}}^{\mathrm{cf}}(\mathbf{u}_{0,h},\mathbf{g}_h)$

$$\begin{aligned}\mathbf{u}_h^{\mathrm{cf}}(0) &= \sum_{n=0}^{N}\sum_{\mathbf{c}\in\mathcal{C}_h}\psi_{n,\mathbf{c}}\mathbf{u}_{0,h} = \mathbf{u}_{0,h} \qquad \text{in } \Omega\,,\\ \big(\underline{A}_{\mathbf{n}}\mathbf{u}_h^{\mathrm{cf}}\big)_k &= \sum_{n=0}^{N}\sum_{\mathbf{c}\in\mathcal{C}_h}\psi_{n,\mathbf{c}}g_{k,h} = g_{k,h} \qquad \text{on } (0,T)\times\Gamma_k,\ k=1,\dots,m\,,\end{aligned}$$

so that for all $\mathbf{z}\in\mathcal{V}^*$ integration by parts and the boundary conditions in $\mathcal{V}^*$ gives

$$\begin{aligned}\big(\mathbf{u}_h^{\mathrm{cf}},L^*\mathbf{z}\big)_Q &= \big(L\mathbf{u}_h^{\mathrm{cf}},\mathbf{z}\big)_Q + \big(M\mathbf{u}_h^{\mathrm{cf}}(0),\mathbf{z}(0)\big)_\Omega - \big(\underline{A}_{\mathbf{n}}\mathbf{u}_h^{\mathrm{cf}},\mathbf{z}\big)_{(0,T)\times\partial\Omega}\\ &= \big(L\mathbf{u}_h^{\mathrm{cf}}-\mathbf{f}_h,\mathbf{z}\big)_Q + \langle \ell_h,\mathbf{z}\rangle\,.\end{aligned}$$

Since $\mathrm{L}_2(Q;\mathbb{R}^m)=M^{-1}L^*(V^*)$ and $\mathcal{V}^*\subset V^*$ is dense, we obtain by duality

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h^{\mathrm{cf}}\|_W &= \sup_{\mathbf{v}\in\mathrm{L}_2(Q;\mathbb{R}^m)\setminus\{\mathbf{0}\}}\frac{\big(M(\mathbf{u}-\mathbf{u}_h^{\mathrm{cf}}),\mathbf{v}\big)_Q}{\|\mathbf{v}\|_W} = \sup_{\mathbf{z}\in\mathcal{V}^*\colon L^*\mathbf{z}\neq\mathbf{0}}\frac{\big(\mathbf{u}-\mathbf{u}_h^{\mathrm{cf}},L^*\mathbf{z}\big)_Q}{\|M^{-1}L^*\mathbf{z}\|_W}\\ &= \sup_{\mathbf{z}\in\mathcal{V}^*\colon L^*\mathbf{z}\neq\mathbf{0}}\frac{\big(L\mathbf{u}_h^{\mathrm{cf}}-\mathbf{f}_h,\mathbf{z}\big)_Q+\langle\ell-\ell_h,\mathbf{z}\rangle}{\|L^*\mathbf{z}\|_{W^*}}\\ &\le \sup_{\mathbf{z}\in\mathcal{V}^*\colon L^*\mathbf{z}\neq\mathbf{0}}\frac{\big\|L\mathbf{u}_h^{\mathrm{cf}}-\mathbf{f}_h\big\|_{W^*}\big\|\mathbf{z}\big\|_W}{\|L^*\mathbf{z}\|_{W^*}} + \sup_{\mathbf{z}\in\mathcal{V}^*\colon L^*\mathbf{z}\neq\mathbf{0}}\frac{\langle\ell-\ell_h,\mathbf{z}\rangle}{\|L^*\mathbf{z}\|_{W^*}}\\ &\le 2T\,\big\|L\mathbf{u}_h^{\mathrm{cf}}-\mathbf{f}_h\big\|_{W^*} + \beta^{-1}\sup_{\mathbf{z}\in\mathcal{V}^*\colon L^*\mathbf{z}\neq\mathbf{0}}\frac{\langle\ell-\ell_h,\mathbf{z}\rangle}{\|\mathbf{z}\|_{V^*}}\end{aligned}$$

using the a priori estimate $\|\mathbf{z}\|_W\le 2T\,\|L^*\mathbf{z}\|_{W^*}$ from Rem. 16 with $C_L=2T$ and $\|\mathbf{z}\|_{V^*}\le\beta^{-1}\|L^*\mathbf{z}\|_{W^*}$ with $\beta^{-1}=\sqrt{1+4T^2}$ from Cor. 20, so that

$$\|\mathbf{u}-\mathbf{u}_h\|_W \le \|\mathbf{u}-\mathbf{u}_h^{\mathrm{cf}}\|_W + \|\mathbf{u}_h^{\mathrm{cf}}-\mathbf{u}_h\|_W$$

yields the assertion. $\square$

This lemma shows that the corresponding error estimator with local contributions

$$\eta_{n,K} = \Big( \sum_{\mathbf{c}\in\mathcal{C}_K} \eta_{n-1,\mathbf{c}}^2 + \eta_{n,\mathbf{c}}^2 \Big)^{1/2}, \qquad \eta_{n,\mathbf{c}} = \Big( \big\| M^{1/2}(\psi_{n,\mathbf{c}}\mathbf{u}_h - \mathbf{u}_{n,\mathbf{c}}^{\mathrm{cf}}) \big\|_{Q_{n,\mathbf{c}}}^2 + 2T \, \big\| M^{-1/2}(L\mathbf{u}_h^{\mathrm{cf}} - \mathbf{f}_h) \big\|_{Q_{n,\mathbf{c}}}^2 \Big)^{1/2},$$

is reliable up to the data approximation error, i.e.,

$$\big\| \mathbf{u} - \mathbf{u}_h \big\|_W \le \Big( \sum_{n=0}^N \sum_{K\in\mathcal{K}_h} \eta_{n,K}^2 \Big)^{1/2} + \beta^{-1} \sup_{\mathbf{z}\in\mathcal{V}^*\setminus\{\mathbf{0}\}} \frac{\langle \ell - \ell_h, \mathbf{z}\rangle}{\|\mathbf{z}\|_{V^*}}\,.$$

**Bibliographic comments.** This chapter is based on [Dörfler et al., 2016, Dörfler et al., 2019], where also numerical results for the adaptive algorithm are presented. Further applications and several numerical applications are reported in [Findeisen, 2016, Ziegler, 2019, Dörfler et al., 2020].
The extension to estimates for weak solutions is based on the construction of a right-inverse as it is done in [Ern and Guermond, 2016] for conforming Petrov–Galerkin approximations in reflexive Banach spaces.
The estimate for the Legendre polynomials can also be obtained recursively using [Abramowitz and Stegun, 1964, Lem. 8.5.3], see, e.g., [Dörfler et al., 2016, Lem. 8].
The error estimation based on dual-weighted residuals transfers the approach in [Becker and Rannacher, 2001] to our space-time framework, and for the general concepts on error estimation by conforming reconstructions we refer to [Ern and Vohralík, 2015].
The results are closely related to the analysis of space-time discontinuous Galerkin methods for acoustics in [Moiola and Perugia, 2018, Bansal et al., 2021, Imbert-Gérard et al., 2020]. Alternative concepts for space-time discretizations for wave equations are collected in [Langer and Steinbach, 2019]. See also the results in [Banjai et al., 2017, Gopalakrishnan et al., 2017] and more recently in [Perugia et al., 2020, Steinbach and Zank, 2020], and the references therein.

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