Karlsruhe Institute of Technology

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Wavelets
Winter Semester 2013/2014

Problem Sheet 9 of December 23, 2013

Exercise 26:
Let $v \in L^2(\mathbb{R})$ satisfy

$$v(x) = \sqrt{a} \sum_{k \in \mathbb{Z}} m_k v(ax - k)\,,$$

where $a > 0$ and $\{m_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. Show that there exists a 2π-periodic function M such that

$$\widehat{v}(\omega) = M\Big(\frac{\omega}{a}\Big)\,\widehat{v}\Big(\frac{\omega}{a}\Big)\,.$$

Determine the function M explicitly.

Exercise 27:
Let $v \in \ell^1(\mathbb{Z})$ and $w \in \ell^p(\mathbb{Z})$, $p \geq 1$. Verify that the convolution product $v \star w$, defined by

$$(v \star w)_k = \sum_{n \in \mathbb{Z}} v_{k-n} w_n\,,$$

satisfies

$$\|v \star w\|_{\ell^p(\mathbb{Z})} \leq \|v\|_{\ell^1(\mathbb{Z})} \|w\|_{\ell^p(\mathbb{Z})}\,.$$

Exercise 28:
Let the $\ell^2(\mathbb{Z})$ endomorphisms R, $S^{\downarrow}$, and $S^{\uparrow}$ be defined as follows:

- *Reverse ordering*: $(Rv)_k = v_{-k}\,,$
- *Down-sampling by* 2: $(S^{\downarrow}v)_k = v_{2k}\,,$
- *Up-sampling by* 2: $(S^{\uparrow}v)_k = \begin{cases} v_r\,, & k = 2r\,,\\ 0\,, & \text{otherwise}\,. \end{cases}$

Moreover, let $\mathcal{H}\colon \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be given by $(\mathcal{H}v)_k = \sum_{n\in\mathbb{Z}} h_{n-2k} v_n$, where $h \in \ell^1(\mathbb{Z})$.

(a) Show that for $v \in \ell^2(\mathbb{Z})$

$$\mathcal{H}v = S^{\downarrow}(Rh \star v) \quad \text{and} \quad \mathcal{H}^* v = h \star S^{\uparrow} v\,.$$

(b) For $v \in \ell^1(\mathbb{Z})$ we introduce the Fourier series

$$\widehat{v}(\omega) := \sum_{k\in\mathbb{Z}} v_k \exp(ik\omega)\,.$$

Prove the following relations for $v, w \in \ell^1(\mathbb{Z})$:

$$\begin{aligned}
\widehat{v \star w}(\omega) &= \widehat{v}(\omega)\widehat{w}(\omega)\,,\\
\widehat{S^{\downarrow}v}(\omega) &= \frac{1}{2}\Big(\widehat{v}\Big(\frac{\omega}{2}\Big) + \widehat{v}\Big(\frac{\omega}{2} + \pi\Big)\Big)\,,\\
\widehat{S^{\uparrow}v}(\omega) &= \widehat{v}(2\omega)\,,\\
\widehat{Rv}(\omega) &= \overline{\widehat{v}(\omega)} \quad \text{for real-valued } v\,.
\end{aligned}$$

These excersises are discussed in the problem class on **Thursday, January 09, 2014.**