Karlsruhe Institute of Technology

Prof. Dr. Andreas Rieder
Dipl.-Math. techn. Tim Kreutzmann
Institute for Applied and Numerical Analysis
Department of Mathematics

Wavelets
Winter Semester 2013/2014

Problem Sheet 3 of November 11, 2013

Exercise 9:
For $\varphi = \chi_{[-1,1]}$, the indicator function of $[-1,1]$, let the operator $\widetilde{W}_\varphi \colon L^2(\mathbb{R}) \to L^2(\mathbb{R}^2, a^{-2}\,\mathrm{d}a\,\mathrm{d}b)$ be defined via

$$\widetilde{W}_\varphi f(a,b) = \int_{\mathbb{R}} f(t)\big(T^b D^a \varphi\big)(t)\,\mathrm{d}t\,.$$

Construct a piecewise constant function

$$\psi = \sum_{k=-2}^{1} a_k \chi_{[k,k+1]}$$

such that $\widetilde{W}_\psi^* \widetilde{W}_\varphi = c_{\varphi,\psi}\mathrm{Id}$ with a suitable constant $c_{\varphi,\psi} \neq 0$.

Exercise 10:
Let

$$W_\psi f(a,b) = \sqrt{\frac{2}{c_\psi}} \int_{\mathbb{R}} f(t)\big(T^b D^a \psi\big)(t)\,\mathrm{d}t\,,$$

where ψ is a real-valued wavelet.[1] Show that

$$\langle W_\psi f, W_\psi g\rangle_{L^2\left([0,\infty[\times\mathbb{R}, a^{-2}\,\mathrm{d}a\,\mathrm{d}b\right)} = \langle f, g\rangle_{L^2(\mathbb{R})}\,.$$

Deduce from this identity the reconstruction formula

$$f(t) = \frac{2}{c_\psi}\int_0^\infty \int_{\mathbb{R}} \langle f, T^b D^a\psi\rangle_{L^2(\mathbb{R})}\big(T^b D^a \psi\big)(t) a^{-2}\,\mathrm{d}b\,\mathrm{d}a\,.$$

Exercise 11:
Let $g \in L^2(\mathbb{R}^2, a^{-2}\,\mathrm{d}a\,\mathrm{d}b)$ such that there exist $\varepsilon > 0$, $s > 1$ and $h \in L^2(\mathbb{R})$ with

$$\big|g(a,b)\big| \leq |a|^s h(b) \quad \text{for } |a| \leq \varepsilon\,.$$

Show that the following integral, which represents the adjoint wavelet transform w.r.t. the wavelet $\psi \in L^2(\mathbb{R})$, exists in the classical sense:

$$W_\psi^* g(t) = c_\psi^{-1/2}\int_{\mathbb{R}}\int_{\mathbb{R}} g(a,b)\big(T^b D^a \overline{\psi}\big)(t) a^{-2}\,\mathrm{d}a\,\mathrm{d}b\,.$$

Hint: Split the integral over the scale into a small-scale part ($|a| \leq \varepsilon$) and a large-scale part ($|a| > \varepsilon$).

These excersises are discussed in the problem class on **Thursday, November 14, 2013**.

[1] Observe the different normalization of W_ψ.