Karlsruher Institut für Technologie

Department of Mathematics
Institute for Applied and
Numerical Mathematics

JProf. Dr. Katharina Schratz
Dipl-Math. techn. Patrick Krämer

Splitting Methods — Exercise Sheet 7 December 19, 2016

On this exercise sheet our aim is to construct a third order splitting method $\Psi_h^{(m)}$ for the model problem

$$\dot{y} = f^{[1]}(y) + f^{[2]}(y), \quad y(0) = y_0 \tag{1}$$

with flows $\varphi_t^{[1]}$ and $\varphi_t^{[2]}$ corresponding to $f^{[1]}$ and $f^{[2]}$ respectively. We define

$$\Psi_h^{(j)} = \varphi_{b_j h}^{[2]} \circ \varphi_{a_j h}^{[1]} \circ \Psi^{(j-1)}, \quad \Psi^{(0)} = Id.$$

In the lecture we proved the theorem:

The method $\Psi_h^{(m)}$ is consistent of order p if $c_{1,m}^1 = c_{2,m}^1 = 1$ and $c_{\ell,m}^k = 0$ for $k = 2, \dots, p$ and all ℓ.

We found the following recurrences, $c_{\ell,0}^k = 0, \quad \forall\, \ell, k$ and for $j \geq 1$:

$$\begin{aligned}
c_{1,j}^1 &= c_{1,j-1}^1 + a_j, \\
c_{2,j}^1 &= c_{2,j-1}^1 + b_j, \\
c_{1,j}^2 &= c_{1,j-1}^2 + a_j b_j + c_{1,j-1}^1 b_j - c_{2,j-1}^1 a_j, \\
c_{1,j}^3 &= c_{1,j-1}^3 + a_j^2 b_j + 2c_{1,j-1}^1 a_j b_j - 3c_{1,j-1}^2 a_j + \left(c_{1,j-1}^1\right)^2 b_j - c_{1,j-1}^1 c_{2,j-1}^1 a_j + c_{2,j-1}^1 a_j^2, \\
c_{2,j}^3 &= c_{2,j-1}^3 + a_j b_j^2 - 4c_{2,j-1}^1 a_j b_j + 3c_{1,j-1}^2 b_j + \left(c_{2,j-1}^1\right)^2 a_j - c_{1,j-1}^1 c_{2,j-1}^1 b_j + c_{1,j-1}^1 b_j^2.
\end{aligned}$$

Now we are looking for a method

$$\Psi_h^{(3)}(y) = \varphi_{b_3 h}^{[2]} \circ \varphi_{a_3 h}^{[1]} \circ \varphi_{b_2 h}^{[2]} \circ \varphi_{a_2 h}^{[1]} \circ \varphi_{b_1 h}^{[2]} \circ \varphi_{a_1 h}^{[1]}(y),$$

where we choose the coefficients $a_j, b_j,\ j = 1,2,3$ such that $\Psi_h^{(3)}$ is of order $p = 3$, i.e. by the theorem above $a_j, b_j,\ j = 1,2,3$ have to satisfy

$$c_{1,3}^1 = 1, \quad c_{2,3}^1 = 1, \quad c_{1,3}^2 = 0, \quad c_{1,3}^3 = 0, \quad c_{2,3}^3 = 0. \tag{2}$$

Exercise 15: (Third Order Splitting Scheme)

a) ♣ Determine the expression for the coefficients $c_{\ell,3}^k$ from equation (2) in terms of $a_j, b_j, j = 1,2,3$, e.g. $c_{1,3}^1 = a_1 + a_2 + a_3$.

b) Determine the coefficients $a_j, b_j,\ j = 1,2,3$ such that (2) is satisfied.

Hint: *For the coefficients we obtain a nonlinear system of equations. Since we have 6 unknowns but only 5 equations there will be many solutions. We can reduce the degrees of freedom by fixing one of the coefficients to a particular value. For example setting $a_1 = 1$, we find a simple solution.*

c) MATLAB : Implement and test the method Ψ_h^3 (with the coefficients of part b)) applied to problem (1), where we assume $f^{[1]}$ and $f^{[2]}$ to be linear, i.e. $f^{[1]}(y) = Ay,\ f^{[2]}(y) = By$ for random complex matrices $A, B \in \mathbb{C}^{N \times N}$, $N = 20$. We furthermore choose random initial data $y_0 \in \mathbb{C}^N$. Create an order plot for $\Psi_h^{(3)}$ by comparing the numerical with the exact solution for various step sizes h.

Exercise 16: (Strang Splitting)

Verify that the coefficients of the Strang splitting method satisfy the order conditions from above up to $p = 2$.

Discussion in the problem class Thursday 15:45, in room 2.067 in the Kollegiengebäude Mathematik (20.30).

♣ : Please try to do exercises marked with ♣ at home.