Karlsruher Institut für Technologie

Department of Mathematics
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**Splitting Methods — Exercise Sheet 6** November 30, 2016

Consider the differential equation

$$\dot{y} = f^{[1]}(y) + f^{[2]}(y), \quad y(0) = y_0.$$

with $f^{[1]}(y) = y^2, \quad f^{[2]}(y) = 1.$

**Exercise 13:** (Lemma by Gröbner)

a) Verify that the flows $\varphi^{[j]}$ corresponding to $f^{[j]},\ j = 1, 2$ are given by

$$\varphi_s^{[1]}(y_0) = \frac{y_0}{1 - s y_0}, \qquad \varphi_t^{[2]}(y_0) = t + y_0.$$

b) Check that the Lemma by Gröbner applies to this example, i.e. show that

$$\left(\varphi_s^{[1]} \circ \varphi_t^{[2]}\right)(y_0) = \exp(tD_2)\exp(sD_1)Id(y_0),$$

where $D_j$ is the Lie derivative corresponding to $f^{[j]}$.
**Hint:** In the lecture we showed the representation

$$\varphi_t^{[j]}(y) = \exp(tD_j)Id(y).$$

c) Show by induction that for the flows $\varphi_{s_j}^{[j]},\ j = 1, \ldots, m$ corresponding to the differential equation

$$\dot{y} = f^{[1]}(y) + f^{[2]}(y) + \cdots + f^{[m]}(y), \quad y(0) = y_0$$

we have

$$\left(\varphi_{s_m}^{[m]} \circ \cdots \circ \varphi_{s_2}^{[2]} \circ \varphi_{s_1}^{[1]}\right)(y_0) = \exp(s_1 D_1)\exp(s_2 D_2)\cdots\exp(s_m D_m)Id(y_0)$$

**Hint:** Use that for $F$ smooth enough we have that

$$F\left(\varphi_t^{[i]}(y_0)\right) = \exp(tD_i)F(y_0).$$

**Exercise 14:**

Let $f^{[1]}(y)$ and $f^{[2]}(y)$ be defined on an open set.

With the help of the BCH formula, show that the corresponding flows $\varphi_s^{[1]}$ and $\varphi_t^{[2]}$ commute if and only if $[D_1, D_2] = 0$.

Discussion in the problem class Thursday 15:45, in room 2.067 in the Kollegiengebäude Mathematik (20.30).

♣ : Please try to do exercises marked with ♣ at home.