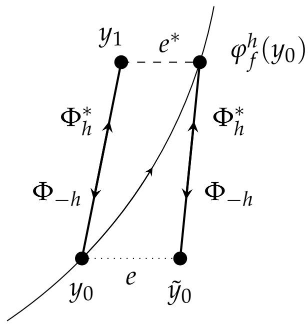
Karlsruher Institut für Technologie
Department of Mathematics
Institute for Applied and
Numerical Mathematics

JProf. Dr. Katharina Schratz
Dipl-Math. techn. Patrick Krämer

Splitting Methods — Exercise Sheet 3 October 28, 2016

On this exercise sheet we want to get familiar with the definition of the adjoint of a method.
We consider the ODE



$$\dot{y} = f(y) = f_1(y) + f_2(y), \quad y(0) = y_0 \tag{1}$$

with exact flow φ_h^f.
The *exact* flow φ_h of a differential equation satisfies $\varphi_h = \varphi_{-h}^{-1}$ but this identity in general does not hold for a numerical method Φ_h.
The **adjoint** method Φ_h^* of Φ_h is given by

$$\Phi_h^* := \Phi_{-h}^{-1}.$$

In other words $y_1 = \Phi_h^*(y_0)$ is implicitly defined by $\Phi_{-h}(y_1) = y_0$.
A method Φ_h is called *symmetric* if $\Phi_h^* = \Phi_h$.

Exercise 6: (Order of the adjoint method, cf. [Hairer et al., 2001, Chapter II.3])

Let Φ_h be a method of order p, i.e

$$\Phi_h(y_0) = \varphi_h^f(y_0) + C(y_0)h^{p+1} + \mathcal{O}\left(h^{p+2}\right).$$

a) Show that the adjoint method of the explicit Euler method is the implicit Euler method and vice versa.

b) ♣ Show that the adjoint method Φ_h^* is of same order p and that there holds

$$\Phi_h^*(y_0) = \varphi_h^f(y_0) + (-1)^p C(y_0)h^{p+1} + \mathcal{O}\left(h^{p+2}\right).$$

Hint: Consider the local error $e^* = y_1 - \varphi_h^f(y_0)$ of Φ_h^* and project it back to the local error e of Φ_{-h}. (see figure).

c) Show that the order p of a symmetric method is even, i.e. $p = 2n, n \in \mathbb{N}$.

Exercise 7: (Composition with the adjoint method, cf. [Hairer et al., 2001, Chapter II.4])

Let Φ_h and Φ_h^* respectively be a numerical method of order p.

a) ♣ Show that for $s > 0$ the composite method

$$\Psi_h = \Phi_{\alpha_s h} \circ \Phi_{\beta_s h}^* \circ \cdots \circ \Phi_{\beta_2 h}^* \circ \Phi_{\alpha_1 h} \circ \Phi_{\beta_1 h}^*$$

has order $p+1$ if

$$\sum_{j=1}^{s} \alpha_j + \beta_j = 1 \qquad \text{and} \qquad \sum_{j=1}^{s} \alpha_j^{p+1} + (-1)^p \beta_j^{p+1} = 0.$$

Let $\varphi_h^{f_1}$ and $\varphi_h^{f_2}$ be the exact flows of the subproblems of (1). We define the Lie splitting method by

$$\Phi_h = \varphi_h^{f_1} \circ \varphi_h^{f_2}.$$

(b) Why is $\varphi_h = \varphi_{-h}^{-1}$ for the exact flow of a differnetial equation? Show that $\Phi_h^* = \varphi_h^{f_2} \circ \varphi_h^{f_1}$.

(c) Show that the Strang splitting method $\Psi_h = \varphi_{h/2}^{f_1} \circ \varphi_h^{f_2} \circ \varphi_{h/2}^{f_1}$ is of order $p = 2$.
Hint: Use part a).

(d) Is the Strang splitting method symmetric? Use that $(\tilde{\Phi}_h \circ \tilde{\Psi}_h)^* = \tilde{\Psi}_h^* \circ \tilde{\Phi}_h^*$ (why?).

Discussion in the problem class Thursday 15:45, in room 2.067 in the Kollegiengebäude Mathematik (20.30).

♣ : Please try to do exercises marked with ♣ at home.
Reference: *Hairer, Lubich, Wanner: Geometric Numerical Integration, 2nd Edition. Springer, 2006*