Introduction to harmonic analysis with applications to evolution equations

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Lecture 2

I L.P. Theory

① Fourier transform

$$\hat{f}(\xi) = \int e^{-ix\xi} f(x)\,dx$$

Characters of $\mathbb{R}^d$

group homom. $\mathbb{R}^d \longrightarrow S^1$

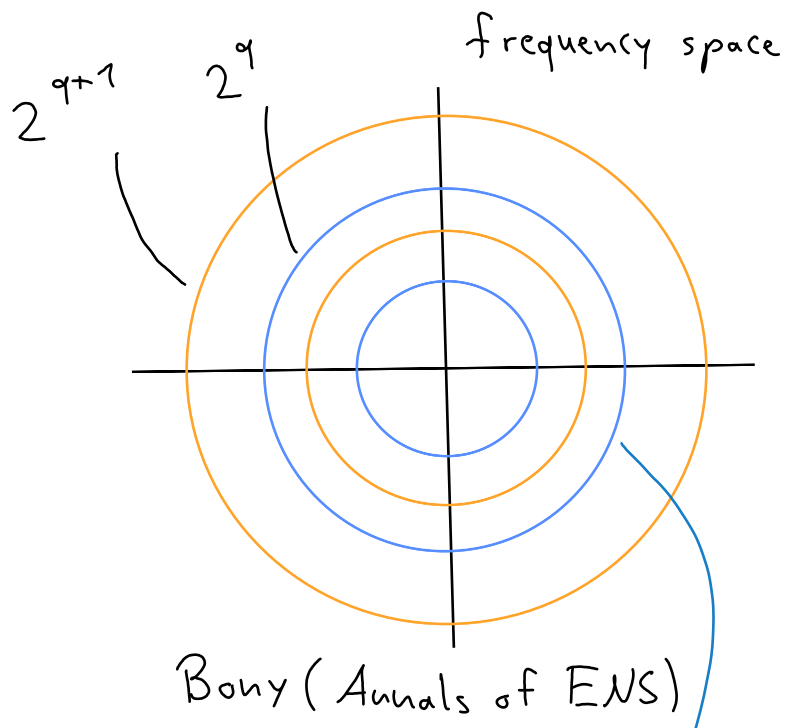
$$\mathcal{F}(f * g) = \mathcal{F}(f)\,\mathcal{F}(g)$$

Symmetry properties $(\mathbb{R}^d)$

$\lambda > 0$ $\quad u_\lambda(x) = u(\lambda x)$

$$\hat{u}_\lambda(\xi) = \lambda^{-d}\,\hat{u}(\xi/\lambda)$$

* Scaling
* reduce the study



$\mathrm{Supp}\,\varphi(2^{-q}\xi) \subset 2^q \mathcal{C}$

2/ $\mathcal{F}$ commutes with the rotations

* $u(x) = g(Rx) \Rightarrow \hat{u}(x) = \hat{g}(Rx)$
  (R: rotation)

* $L^2 = \bigoplus_{k \in \mathbb{Z}} D^k$ — invariant under $\mathcal{F}$

(Stein-Weiss) chap. 4

$L^2(\mathbb{R})$, $g(x) = g(re^{i\theta})$

$$g(x) = \sum_{k=-\infty}^{\infty} e^{ik\theta} f_k(r)$$

$$D^k = \{h \mid h(x) = e^{ik\theta} f(r)\}$$

$\mathcal{F} : D^k \longrightarrow D^k$

② L.P. decomposition

Proposition (dyadic unity partition, B.C.D)

Let $\mathcal{C} = \{\xi \in \mathbb{R}^d \mid 3/4 \leq |\xi| \leq 8/4\}$.

Then $\exists\, \varphi \in \mathcal{D}(\mathcal{C})$ and $\chi \in \mathcal{D}(B(0, 4/3))$ radial, valued in $[0,1]$, and

1) $\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q}\xi) = 1, \quad \forall \xi$

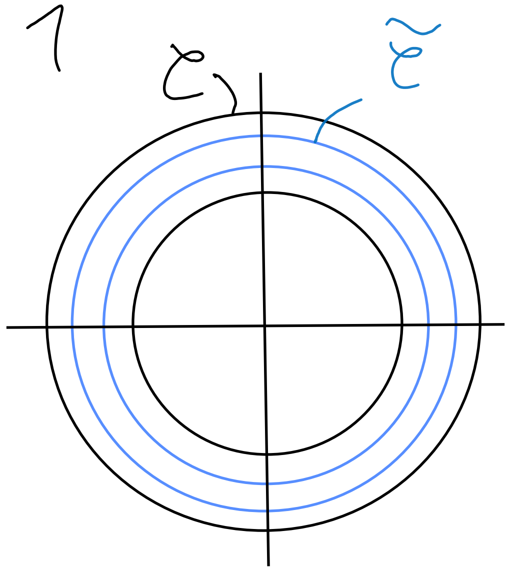
2) $\sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1, \quad \forall \xi$

3) $\boxed{\begin{array}{l} |j-j'| \geq 2 \Rightarrow \operatorname{supp} \varphi(2^{-j}\xi) \cap \operatorname{supp} \varphi(2^{-j'}\xi) = \emptyset \\ j \geq 1 \Rightarrow \operatorname{supp} \chi \cap \operatorname{supp} \varphi(2^{-j}\xi) = \emptyset \end{array}}$

4) "Quasi-orthogonality"

$$\tfrac{1}{2} \leq \chi^2(\xi) + \sum_{q \geq 0} \varphi^2(2^{-q}\xi) \leq 1$$

$$\tfrac{1}{2} \leq \sum_{q \in \mathbb{Z}} \varphi^2(2^{-q}\xi) \leq 1$$



Θ radial function, $\operatorname{supp} \subset \mathscr{C}$

$\Theta = 1$ in $\tilde{\mathscr{C}}$ (= smaller annulus)

$$S(\xi) = \sum \Theta(2^{-q}\xi) \quad | \text{ Locally finite}$$

$$\varphi(\xi) = \frac{\Theta(\xi)}{S(\xi)}.$$

$$\left(\chi(\xi) + \sum \varphi(2^{-q}\xi)\right)^2 = 1$$

$$\Rightarrow \chi^2(\xi) + \sum \varphi(2^{-q}\xi) \leq 1$$

$$\left(\sum_{\text{odd}} + \sum_{\text{even}}\right)$$

$\rightsquigarrow$ We can split any function into a series of regular functions with frequencies in an annulus of size $\sim 2^q$.

<u>Notations</u>

<u>inhom.</u> $\Delta_q u = \mathcal{F}^{-1}(\varphi(2^{-q}\cdot)\hat{u})$, Spectr. $\subset 2^q \mathscr{C}$

<u>from</u> $\dot{\Delta}_q u$

* $u \overset{?}{=} \sum \Delta_q u$ (the inhomogeneous case in $\mathcal{S}'$)
* homogeneous case:

non-convergence to u for polynomials.

$(\mathcal{S}'_h)$

"in some sense" $u = \sum \underbrace{\Delta_q u}_{\mathcal{F}^{-1}(\varphi(2^{-q}\xi)\hat{u}(\xi))}$

• $\underbrace{\Delta_q u}$: truncate in frequency ($\simeq 2^q$) the function u.

$\rightsquigarrow \Delta_q u$ is a convolution operator

$$\Delta_q u = h_q * u$$

$$= 2^{qd} h(2^q \cdot)$$

$\Delta_q : L^p \longrightarrow L^p$ continuous with norm indep. of p and q.

$$\|\Delta_q u\|_{L^p} \le \underbrace{\|h_q\|_{L^1}}_{=\|h\|_{L^1}} \|u\|_{L^p}$$

Bernstein inequalities

1) $\mathrm{supp}(\hat{u}) \subset \lambda\, \mathcal{C}$

$$\|D^\alpha u\|_{L^p} \sim \lambda^{|\alpha|} \|u\|_{L^p}$$

$$\hat{u}(\xi) = \Phi(\lambda^{-1}\xi)\, \hat{u}(\xi)$$

* Scale invariance: reduce to the unit ring.

$$u = h * u \qquad (h \in \mathcal{S})$$

$$Du = Dh * u$$

$$\|D^\alpha u\|_{L^p} \le \|D^\alpha h\|_{L^1} \|u\|_{L^p}$$

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converse inequality:

$$|\xi|^{2k} = (|\xi|^2)^k = (\xi_1^2 + \dots + \xi_d^2)^k$$

$$= \sum_{|\beta|=k} A_\beta (i\xi)^\beta (-i\xi)^\beta.$$

$\lambda = 1$ $\quad \|u\|_{L^p} \le \|D^\alpha u\|_{L^p} \qquad (\mathrm{supp}(\hat{u}) \subset \mathcal{C})$

$$\hat{u}(\xi) = \frac{|\xi|^{2k}}{|\xi|^{2k}} \Phi(\xi)\hat{u}(\xi), \qquad \Phi \in \mathcal{D}(\mathbb{R}^d \setminus \{0\}), \quad \Phi = 1 \text{ near } \mathcal{C}$$

$$= \frac{\sum_{|\beta|=k} A_\beta (i\xi)^\beta \Phi(\xi) \widehat{D^\alpha u}(\xi)}{|\xi|^{2k}}$$

$$= \sum_{|\beta|=k} g_\beta(\xi)\, \widehat{D^\beta u}(\xi)$$

$$\Longrightarrow u = \sum h_\beta * D^\beta u$$

$$\|u\|_{L^p} \le \sum \|h_\beta\|_{L^1} \|D^\beta u\|_{L^p}.$$

1) $\|u\|_{H^s} \sim \left\| \left(2^{qs} \|\Delta_q u\|_{L^2}\right) \right\|_{\ell^2_q}$

Quasi-orthogonality:

$$\|u\|_{\dot{H}^s}^2 \sim \int |\xi|^{2s} |\hat{u}(\xi)|^2\, d\xi$$

$$u = \sum \Delta_q u, \quad \hat{u} = \sum \widehat{\Delta_q u} = \sum \varphi(2^{-q}\xi)\hat{u}(\xi)$$

$$\left(\left(\sum \varphi(2^{-q}\xi)\right)^2 \sim \sum \varphi^2(2^{-q}\xi) \right)$$

$$\Longrightarrow \|u\|_{\dot{H}^s}^2 \sim \sum_q \int |\xi|^{2s} \varphi^2(2^{-q}\xi) |\hat{u}(\xi)|^2\, d\xi$$

$$\Rightarrow \|u\|^2_{\dot H^s} \sim \sum 2^{2qs} \underbrace{\int \varphi^2(2^{-q}\xi)\,|\hat u(\xi)|^2\,d\xi}_{=\|\Delta_q u\|_{L^2}}$$

$$\underbrace{\|u\|_{H^s}}_{B^s_{2,2}} = \Big\| \big(2^{qs}\,\|\Delta_q u\|_{L^2}\big)_q \Big\|_{\ell^2_q}$$

regularity 2 2

Besov spaces: $B^s_{p,r}$

$$\|u\|_{B^s_{p,r}} = \Big\| \big(2^{qs}\,\|\Delta_q u\|_{L^p}\big)_q \Big\|_{\ell^r}$$

Note: r is a „regularity fine tuning parameter":
- $B^s_{p,1} \subseteq B^s_{p,r} \subseteq B^s_{p,\infty}$
- $B^{s+\varepsilon}_{p,r_1} \subseteq B^s_{p,r_2}$ $\quad \forall r, r_1, r_2 \in [1,\infty]$

Bernstein $\Rightarrow$ study sobolev embedding.

Bony decomposition

$$\text{"}uv\text{"} = \sum_{p,q} \Delta_p u\, \Delta_q v$$

$$= \sum_{p\le q-2} \Delta_p u\, \Delta_q v + \sum_{q \le p-2} \Delta_p u\, \Delta_q v$$

Paraproduct terms

$$+ \sum_{|p-q|\le 1} \Delta_p u\, \Delta_q u$$

Remainder term

supp $\subseteq 2^q B$

II. Dispersion Phenomena for evolution equations

(E) evolution equation and u

$$\|u(t,\cdot)\|_{L^\infty} \lesssim \frac{1}{|t|^{r}} \|u_0\|_{L^1} \qquad r > 0$$

(family of estimates)

(S) $$u(t,x) = C \int e^{ix\xi + it|\xi|^2}\, \hat u_0(\xi)\, d\xi$$

$$\frac{1}{|t|^{d/2}}$$

(W) $$v(t,x) = v^+(t,x) + v^-(t,x)$$

$$v^\pm(t,x) = C_d \int e^{ix\xi \pm it|\xi|}\, \widehat{v^\pm}(\xi)\, d\xi$$

$$\frac{1}{|t|^{\frac{d-1}{2}}}$$

"stationary phase theorems"

$$I_t = \int_{\mathbb{R}^d} e^{it\varphi(x)}\, a(x)\, dx \overset{t\to\infty}{\sim} \;?$$

(oscillating integral)

φ admits a non-degenerate critical point

$$\rightsquigarrow I_t = \frac{C_d}{t^{d/2}} \sum_{n=0}^{N} \frac{C_n}{t^n} + O\left(\frac{1}{|t|^{N+1}}\right)$$

$a(\ldots)$

Lecture 3 Dispersion phenomena

①

$$(S)\begin{cases} i\partial_t u - \Delta u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

$$u(t,x) = (2\pi)^{-d}\int e^{ix\xi + it|\xi|^2}\hat{u}_0(\xi)\,d\xi$$

$$= (S_t * u_0)(x)$$

$$S_t(x) = \frac{1}{(-4i\pi t)^{d/2}}\int e^{-\frac{i|x|^2}{4t}}$$

$$\Longrightarrow \|u(t,\cdot)\|_{L^\infty} \lesssim \frac{1}{|t|^{d/2}}\|u_0\|_{L^1}.$$

$$\mathcal{F}^{-1}(\underbrace{e^{it|\xi|^2}}_{\notin L^1})(x) = S_t(x)$$

①

$$z \longrightarrow H_1(z) = (2\pi)^{-d}\int e^{ix\xi}e^{-z|\xi|^2}d\xi$$

$$z \longrightarrow H_2(z) = \frac{1}{(4\pi z)^{d/2}}e^{-\frac{|x|^2}{4z}}$$

both holomorphic in $D = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$

② Heat kernel

$H_1 = H_2$ on $D \cap \mathbb{R}$ $\Longrightarrow$ $H_1 = H_2$ on D.

③

$$\int H_1(z_n)\underbrace{\phi(x)}_{\in\mathcal{S}}dx \longrightarrow \dots$$

$$\|$$

$$\int H_2(z_n)\phi(x)\,dx \longrightarrow \dots$$

$\|u(t,\cdot)\|_{L^2}^2 = \|u_0\|_{L^2}^2$ (conservation of mass)

$$\int (i\partial_t u - \Delta u)\bar{u} - \overline{(i\partial_t u - \Delta u)}u = 0$$

$$\|$$

$$\frac{d}{dt}\|u(t)\|_{L^2}^2.$$

$$\hat{u}(t,\xi) = e^{it|\xi|^2}\hat{u}_0(\xi)$$

$$\Longrightarrow \|\hat{u}(t,\cdot)\|_{L^2}^2 = \|u_0\|_{L^2}^2$$

$$\xrightarrow{\text{Plancherel}} \|u(t,\cdot)\|_{L^2}^2 = \|u_0\|_{L^2}^2$$

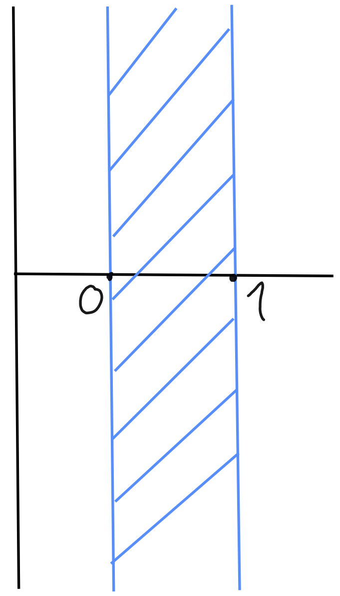
By <u>complex interpolation</u> argument:

$$\|u(t,\cdot)\|_{L^p} \lesssim \frac{\|u_0\|_{L^{p'}}}{|t|^{d/p' - d/2}}$$

Theorem (Riesz-Thorin)

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$$

$$A : L^{p_0}(X,\mu) \longrightarrow L^{q_0}(Y,\nu) \quad \mathcal{A}_0$$
$$\quad : L^{p_1}(X,\mu) \longrightarrow L^{q_1}(Y,\nu) \quad \mathcal{A}_1$$
$$\Longrightarrow$$
$$A : L^{p_\theta}(X,\mu) \longrightarrow L^{q_\theta}(Y,\nu) \quad \mathcal{A}_0^{1-\theta}\,\mathcal{A}_1^{\theta}$$



$(S) \longrightarrow r = d/2$ optimal, thanks to stationary phase theorem.

$$\int_{\mathbb{R}^d} e^{it\boxed{Q(\xi)}} a(\xi) \sim \frac{C}{t^{d/2}} a(0) \qquad \text{(Asymptotic)}$$

non-degenerate quadratic frame

TT* Argument (Ginibre-Velo, Keel-Tao)

Theorem (TT* argument)

$(u(t))_{t\in\mathbb{R}}$ be a bounded family of operators on L^2.

$$\| u(t') u^*(t') f \|_{L^\infty} \leq \frac{C}{|t-t'|^\sigma} \| f \|_{L^1}$$

$$\Longrightarrow \forall (q,r) : \frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{2} \qquad \left(\begin{array}{l} (q,r) \in [2,\infty] \\ (q,r,\sigma) \neq (2,\infty,1) \end{array} \right)$$

$$\| u(t) u_0 \|_{L^q_t(L^r_x)} \lesssim \| u_0 \|_{L^2}$$

Duality arguments

$$\| u(t) u_0 \|_{L^q(L^r)} = \sup_{\varphi \in B_{q',r'}} \left| \int (u(t)u_0)(x)\, \varphi(t,x)\, dt\, dx \right|$$

$$B_{q',r'} = \{ \varphi \in D \mid \|\varphi\|_{L^{q'}(L^{r'})} \leq 1 \}$$

$$\iint (u(t)u_0)(x)\, \varphi(t,x)\, dt\, dx$$
$$= \iint u_0(x)\, u^*(t)\, \overline{\varphi}(t,x)\, dt\, dx$$

$$\Longrightarrow \| u(t) u_0 \|_{L^q(L^r)} \leq \| u_0 \|_{L^2} \sup_{\varphi \in B_{q',r'}} \left\| \int u^*(t)\, \varphi(t,x)\, dt \right\|_{L^2_x}$$

$$\| \dots \|^2_{L^2} = \iint u^*(t)\, \varphi(t,x)\, u^*(t')\, \varphi(t',x)\, dt\, dt'$$

+ Hölder inequality:

$$\leq \iint \left\| \frac{1}{|\cdot|^{2/q}} * \|\varphi(\cdot,-)\|_{L^{q'}} \right\| \varphi(t',-) \|_{L^{q'}}$$

+ Sobolev-Hardy inequality ....

$$(W) \quad \begin{cases} \partial_t^2 u - \Delta u = 0 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases}$$

spectrally localized in a ring

$$\Longrightarrow \hat{u}(t,\xi) = \hat{u}_+(t,\xi) + \hat{u}_-(t,\xi)$$

$$\hat{u}_\pm(t,\xi) = e^{\pm it|\xi|} \underbrace{Y_\pm^1(\xi)}_{= \frac{1}{2}\left(\hat{u}_0(\xi) \mp \frac{\hat{u}_1(\xi)}{|\xi|}\right)}$$

$$u_+(t,x) = k_+^+ * Y_+ \qquad \|Y_+\|_{L^1} \lesssim \left(\|u_0\|_{L^1} + \|u_1\|_{L^1}\right)$$

$$k_t^+(x) = \int e^{ix\xi + it|\xi|} \underbrace{\phi(\xi)}_{\in\, \mathcal{D}(\mathbb{R}^d \setminus \{0\})} d\xi$$

$$\|k_t^+\|_{L^\infty} \lesssim \frac{1}{|t|^{\frac{d-1}{2}}}. \qquad \|u^\pm(t,\cdot)\|_{L^\infty} \lesssim \frac{1}{|t|^{d/2}} \|u_0\|_{L^1}$$

Stationary and non-stationary phase estimate.

$$k_t^+(tx) = \int e^{\overbrace{ix\xi + it|\xi|}^{it\varphi(\xi)}} \phi(\xi)\, d\xi$$

$$\|k_t^+\|_{L^\infty}? \qquad \underbrace{|D_\xi \varphi| \geq 1}_{I_1}, \qquad \underbrace{|D_\xi \varphi| \leq 1/2}_{I_2}$$

I_1: non-stationary phase th

I_2: stationary phase th.

$\mathcal{L}(e^{it\varphi}) = e^{it\varphi}$ + int. by parts.

$$\leadsto I_1 = \int_{\{|D\varphi| \geq \frac{1}{2}\}} e^{it\varphi} \phi(\xi)\, d\xi \Rightarrow \boxed{|I_1(t)| \leq \frac{C_N}{t^N}}$$

$$\mathcal{L} = \frac{-i D_\xi \varphi\, D_\xi}{|D_\xi \varphi|^2} \Rightarrow \mathcal{L}(e^{it\varphi}) = t e^{it\varphi}$$

$$\Rightarrow I_1(t) = \frac{1}{t} \int \mathcal{L}(e^{it\varphi}) \phi(\xi)\, d\xi$$

$$= \frac{1}{t^N} \int_{\{\xi \in \mathcal{C}\}} e^{it\varphi} \left| ({}^t\mathcal{L})^N \phi \right| d\xi \leq C_N.$$

$\hat{}\,\mathcal{D}(\mathcal{C})$

$$I_2(t) = \int_{\{|D\varphi| \leq 1,\ \xi \in \mathcal{C}\}} e^{it\varphi} \phi(\xi)\, d\xi. \qquad \nabla\varphi(\xi) = x + \frac{\xi}{|\xi|}$$

If $|D\varphi| \leq \frac{1}{2} \Rightarrow x \neq 0$

$$\nabla\varphi(\xi) = \underbrace{\left\langle \nabla\varphi \,\middle|\, \frac{x}{|x|} \right\rangle \frac{x}{|x|}}_{a} + \nabla\varphi(\xi) - a$$

$$\mathcal{L}_+ = \frac{\mathrm{Id} - i D_\xi \varphi\, D_\xi}{1 + t|D_\xi \varphi|^2}, \qquad \mathcal{L}_+(e^{it\varphi}) = e^{it\varphi}$$

$$I_2(t) = \int \mathcal{L}_+(e^{it\varphi}) \phi(\xi)\, d\xi$$

$$= \int_{\{\xi \in \mathcal{C}\}} e^{it\varphi} \underbrace{\left[{}^t(\mathcal{L}_+)^{2N} \phi\right]}(\xi)\, d\xi$$

$$|\ldots| \leq \frac{C}{(1+t|D\varphi|^2)^N} \lesssim \frac{C}{(1+t|\xi'|^2)^N}.$$

$$|I_2(t)| \lesssim C_N \int_{\{\xi \in \mathcal{C}\}} \frac{d\xi_1\, d\xi'}{(1+t|\xi'|^2)^N} \qquad \eta = \sqrt{t}\,\xi, \quad \xi = (\xi_1, \xi')$$

$$I_2(t) = \int_{\{|D\varphi| \le \frac{1}{2}\}} \mathcal{L}_+ \left(e^{it\varphi}\right) \Phi(\xi)\, d\xi$$

$$\int_{\{\xi = (\xi_1, \xi') \in \mathcal{C}\}} \frac{d\xi_1\, d\xi'}{(1 + t|\xi'|^2)^N} \lesssim \int \frac{d\xi'}{(1+t|\xi'|)^{2N}} \overset{\eta = \sqrt{t}\xi}{=} \frac{1}{|t|^{d/2-1}} \int_{\mathbb{R}^{d-1}} \frac{d\eta}{(1+|\eta|^2)^N}$$

Jacobian

Note: $|D\varphi| \gtrsim |\xi'|$, $\frac{\xi'}{|\xi|} \in \mathcal{C}$.

Stratified Lie groups

Heisenberg group: $\mathbb{H} = \mathbb{R}^2_{(x_1,x_2)} \times \mathbb{R}_{x_3}$

$(\mathbb{H}, \cdot)$ non-commutative group

$(0,0,x_3)$ center: it commutes for the law

$$(x_1,x_2,x_3) \circ (0,0,y_3) = (0,0,y_3) \circ (x_1,x_2,x_3)$$

Fourier transform tricky!

Characters: $(G,\cdot) \longrightarrow (S^1, \times)$

$$\mathcal{F}(f * g) = \mathcal{F}(f)\,\mathcal{F}(g)$$

Characters of $\mathbb{H} \sim \mathbb{R}^2$

$$\mathcal{F}(f) \overset{?}{=} \int f(x_1,x_2,x_3)\, e^{ix_1\xi_1 + ix_2\xi_2}\, dx_1\, dx_2\, dx_3$$

want good theory: inversion, Plancherel, etc.

$\leadsto$ Fourier transform using unitary representation.

$$f \in L^1(\mathbb{H}), \quad \hat{f}(\lambda) = \int f(x)\, U_x^\lambda\, dx$$

$$U^\lambda : (\mathbb{H}, \cdot) \longrightarrow \underbrace{\mathcal{U}(L^2(\mathbb{R}))}_{\text{unitary operators}}$$

$$\begin{cases} i\partial_t u - \Delta_{\mathbb{H}} u = 0 \\ u|_{t=0} = u_0 \end{cases}$$

harmonic oscillator

$$\mathcal{F}(\Delta_{\mathbb{H}} u)(\lambda) = \hat{U}(\lambda) \circ P_\lambda, \quad P_\lambda = -\frac{d^2}{dt^2} + |\lambda|\,\theta^2$$

Good theory

$$u(t,x) = \sum_m \int_{\{\lambda > 0\}} \underbrace{e^{it|\lambda|(2m+1)} (U^\lambda_x)^*}_{e^{i\lambda((2m+1)t - x_3)}} \mathcal{F}(u_0)(\lambda)\, |\lambda|\, d\lambda$$

$e^{-i\lambda x_3}(\ldots)(x_1,x_2)$

using Fourier inversion formula

both homog. of order 2

$\leadsto$ behaves like transport equ.