Introduction to harmonic analysis with applications to evolution equations

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Introduction

• Main goal : to investigate dispersion phenomena for linear evolution equations with applications to some nonlinear PDEs involved in physics, fluid and quantum mechanics, biology,...

The waves decrease and vanish as the time goes to infinity

- Among the iconic examples of linear evolution equations on $\mathbb{R}^d$, one can mention

- the heat equation : $\partial_t u - \Delta u = 0$

- the transport equation : $\partial_t u + b \cdot \nabla u = 0$

- the Schrödinger equation : $i\partial_t u - \Delta u = 0$

- the wave equation : $\partial_t^2 u - \Delta u = 0$

- One can explicitly solve all these equations which are of different types

Dispersion phenomena express that waves with different frequencies move at different velocities

For instance, for the free linear Schrödinger equation on $\mathbb{R}^d$

$$(S) \qquad \begin{cases} i\partial_t u - \Delta u &= 0 \\ u_{|t=0} &= u_0\,, \end{cases}$$

taking the partial Fourier transform $\mathcal{F}$ of (S) with respect to the variable x, we obtain

$$i\partial_t \mathcal{F}u(\xi) + |\xi|^2 \mathcal{F}u(\xi) = 0.$$

Then integrating in time the resulting ODE, we get

$$\mathcal{F}u(t,\xi) = e^{it|\xi|^2}\mathcal{F}(u_0)(\xi).$$

Applying the inverse Fourier formula, we obtain (oscillating integral)

$$u(t,x) = (2\pi)^{-d}\int e^{i(x\cdot\xi+\mathbf{t|\xi|^2})}\mathcal{F}(u_0)(\xi)d\xi$$

$$= \left(\frac{\mathrm{e}^{i\frac{|\cdot|^2}{4t}}}{(4\pi i t)^{\frac{d}{2}}} \star u_0\right)(x)$$

Commonly dispersive estimates correspond to a pointwise inequality in time decay, namely ($t \neq 0$)

$$\|u(t,\cdot)\|_{L^\infty} \lesssim \frac{\|u_0\|_{L^1}}{|t|^r}$$

where (in general) the rate of decay $r > 0$ depends on the equation, the dimension and the setting

Very often, interpolating such type of estimate with some conservation law, we deduce a family of dispersive inequalities

In the particular case of the free linear Schrödinger equation on $\mathbb{R}^d$, taking advantage of the representation of the solution under convolution form, we deduce thanks to Young inequality

$$\|u(t,\cdot)\|_{L^\infty} \leq \frac{\|u_0\|_{L^1}}{(4\pi|t|)^{\frac{d}{2}}}$$

Since we have the conservation of the mass for the solutions of (S) :

$$\|u(t,\cdot)\|_{L^2}^2 = \|u_0\|_{L^2}^2$$

a complex interpolation argument (Riesz-Thorin's theorem) leads to the following family of dispersive inequalities, for all $1 \leq p \leq 2$

$$\|u(t,\cdot)\|_{L^p} \leq \frac{\|u_0\|_{L^{p'}}}{(4\pi|t|)^{\frac{d}{p}-\frac{d}{2}}}$$

where p' denotes the conjugate exponent of p, namely

$$\frac{1}{p} + \frac{1}{p'} = 1$$

This family of estimates is a key tool in the study of nonlinear Schrödinger equations

Actually a functional analysis argument known as the TT^*-argument initiated by Ginibre-Velo and refined by Keel-Tao enables to deduce from the family of dispersive estimates a bound for the space-time norm of the solution u by the norm of the initial datum u_0 :

$$\|u\|_{L^p_t(L^q_x)} \lesssim \|u_0\|_{L^2}$$

for some suitable (p, q) called admissible pairs.

These pairs which depend on the equation and the dimension can be computed using the scale invariance of the equation

Now in the case of the Schrödinger equation on $\mathbb{R}^d$ (where the rate of decay $r = d/2$), these estimates known as Strichartz estimates are the following

$$\|u\|_{L^p(\mathbb{R},L^q(\mathbb{R}^d))} \leq C\|u_0\|_{L^2(\mathbb{R}^d)}$$

where (p, q) are given by the scaling admissibility condition

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}$$

and satisfy moreover $p \geq 2$ and $(d, p, q) \neq (2, 2, \infty)$.

For instance when $d = 2$, $L^3(L^6)$ is a Strichartz norm.

Note that the Strichartz estimates admit more general forms, whether in the inhomogeneous setting (with a source term f) or in more general functional spaces such as Sobolev or Besov spaces.

Strichartz estimates (which express both a decrease effect and a regularity effect) constitute a central tool in the study of nonlinear equations, whether in semilinear frameworks or in quasilinear settings.

- For instance, if consider the cubic semilinear Schrödinger equation on $\mathbb{R}^2$ (which involves in quantum mechanics) :

$$(NLS_3) \qquad \begin{cases} i\partial_t u - \Delta u &= P_3(u,\overline{u}) \\ u_{|t=0} &= u_0 \in L^2(\mathbb{R}^2)\,, \end{cases}$$

where P_3 is an homogeneous polynomial of degree 3 with respect to u and $\overline{u}$ (for instance, one can take $P_3(u,\overline{u}) = |u|^2 u$), then using that $L^3_t(L^6_x)$ is a Strichartz norm (and then $\|P_3(u,\overline{u})\|_{L^1_t(L^2_x)} \sim \|u\|^3_{L^3_t(L^6_x)}$), one can show thanks to the fixed point theorem that (NLS_3) is locally (in time) wellposed for any Cauchy data in L^2 and even globally (in time) wellposed for small Cauchy data.

More precisely, combining the fixed point theorem together with Strichartz estimates, one can by classical arguments establish the following theorem :

Let $u_0 \in L^2(\mathbb{R}^2)$. Then there exists a positive time $T = T(\|u_0\|_{L^2})$ such that there exists a unique solution u of (NLS_3) in the functional space $C([0,T]; L^2(\mathbb{R}^2)) \cap L^3([0,T]; L^6(\mathbb{R}^2))$. Moreover, there is a positive constant $c > 0$ such that if $\|u_0\|_{L^2} \leq c$, then the solution belongs to $L^3(\mathbb{R}_+; L^6(\mathbb{R}^2)) \cap C_b(\mathbb{R}_+; L^2(\mathbb{R}^2))$.

The defocusing (NLS_3) with $P_3(u, \overline{u}) = |u|^2 u$ is globally wellposed for any Cauchy data in $L^2(\mathbb{R}^2)$, but the proof is much more involved Benjamin Dodson,...

In the focusing case $P_3(u, \overline{u}) = -|u|^2 u$, blow up can occur for large data : wide literature Book of Cazenave.

There is plethora of results in the same vein based on Strichartz estimates for nonlinear Schrödinger equations as well as for nonlinear wave equations on $\mathbb{R}^d$ but also in more general settings, such as on curved manifolds, in the presence of potentials, obstacles, boundary conditions, or in regular variable coefficient situations,...

It would take too long to go through all these results. In my lectures, I will mainly focus on the use of Strichartz estimates to investigate semilinear evolution equations on $\mathbb{R}^d$, but the case of quasilinear evolution equations have been also extensively studied by many authors as well as the case of bilinear estimates

- The use of Strichartz estimates in quasilinear framework such as for the following wave equation in connection with the general relativity

$$(E) \begin{cases} \partial_t^2 u - \Delta u - \partial(g(u)\partial u) &= Q(\nabla u, \nabla u) \\ (u, \partial_t u)_{|t=0} &= (u_0, u_1) \end{cases}$$

for some suitable metric g, is much more involved than in semilinear framework and requires the use of microlocal analysis and in particular the Littlewood-Paley theory and the Paradifferential calculus of J.-M. Bony

The basic tool to prove local solvability for such equation relies on the following energy estimate

$$\|\partial u(t,\cdot)\|_{H^{s-1}} \leq \|\partial u(0,\cdot)\|_{H^{s-1}} e^{\int_0^t \|\partial g(t',\cdot)\|_{L^\infty} dt'}$$

So the key quantity to control is $\int_0^t \|\partial g(t',\cdot)\|_{L^\infty} dt'$

If $(\partial u_0, u_1) \in H^{s-1}$ with $s > d/2 + 1$, then this quantity can be controlled thanks to Sobolev embedding

But the scale invariant space for (E) is $H^{d/2}$: to go below this regularity for the initial data, we have to use the dispersive properties of the wave equation

There are many works devoted to this equation Bahouri-Chemin, Klainerman-Rodnianski-Szeftel, Smith-Tataru, Tataru,.... it is the combination of geometrical optics and harmonic analysis that allows to get closer to the critical space.

Roughly speaking, using the paradifferential calculus of J.-M. Bony, we reduce the issue to the study of u_q the part of the solution relating to frequencies of size 2^q which satisfies a wave equation with regular coefficients, and for which we establish a microlocal Strichartz estimates, namely Strichartz estimates on time intervals whose size depend on the frequency. This is due to the fact that the regular coefficients of the wave equation satisfied by u_q take in mind the starting regularity and cannot be uniformly bounded with respect to 2^q.

To conclude, we glue the microlocal estimates to obtain a local Strichartz estimate and improve the threshold regularity given by the energy estimate

Nowadays, it is well known that the geometry of the setting has a big influence on the dispersion effect, and many authors have focused on these matters. Among others, one can mention the results of

- Anker-Pierfelice, Banica, Pierfelice,... on the real hyperbolic space

- Bourgain on the Torus

- Burq-Gérard-Tzvetkov and Staffilani-Tataru on compact manifolds

- Ivanovici-Lebeau-Planchon,... on some domains

- Banica-Duyckaerts,... on noncompact manifolds

- Bahouri-Gérard, Bahouri-Fermanian-Gallagher, Del Hierro, Furioli-Melzi-Veneruso, Müller-Seeger,... on stratified Lie groups such as the Heisenberg group $\mathbb{H}^d$ which is a step 2 stratified Lie group

In particular there are many situations where dispersion phenomena fail ! (for instance in compact manifolds, on some domains, on the Heisenberg group,...)

However in some cases, Strichartz estimates (in weak forms) or smoothing properties can be established using other approches, which are not based on dispersive inequalities

Among these approches, one can mention the methods based on

- Fourier restriction theorems

- Kato smoothing effect

Fourier restriction theorems

The subject is wide and there are several references and monographs

Tao : Some recent progress on the restriction conjecture, Fourier Analysis and Convexity

Stein : Harmonic Analysis : Real-Variable Methods, Orthogonality, Oscillatory integrals

On $\mathbb{R}^d$, the basic Fourier restriction theorem which is due to Tomas-Stein states as follows :

Let $d \geq 2$ and $p_0 = \frac{2(d+1)}{d+3}$. Let S be a smooth and compact hypersurface in $\mathbb{R}^d$ endowed with a smooth measure $d\sigma$. Assume that S has non vanishing Gaussian curvature at every point. Then for all $p \in [1, p_0]$, there exists a positive constant $C = C_p$ such that for any $f \in \mathcal{S}(\mathbb{R}^d)$:

$$\|(\mathcal{F}f)_{|S}\|_{L^2(S,d\sigma)} \leq C\|f\|_{L^p(\mathbb{R}^d)}$$

The canonical example of S in this theorem is the sphere $\mathbb{S}^{d-1}$

If we denote by R_S the restriction operator $R_S f = \mathcal{F}(f)|_S$, then by Tomas-Stein's theorem it is continuous from $L^p(\mathbb{R}^d)$ to $L^2(S, d\sigma)$. By duality arguments, Tomas-Stein's theorem is equivalent to say that the adjoint operator R_S^*

$$R_S^* g = \mathcal{F}^{-1}(g d\sigma)$$

is continuous from $L^2(S, d\sigma)$ to $L^{p'}(\mathbb{R}^d)$, namely

$$\|\mathcal{F}^{-1}(g d\sigma)\|_{L^{p'}(\mathbb{R}^d)} \leq C\|g\|_{L^2(S,d\sigma)}$$

Tomas-Stein's theorem gives an answer to the following problem : can we restrict the Fourier transform of an L^p function to a subset ?

- When $p = 1$, the answer is obvious since by Riemann-Lebesgue theorem, $\mathcal{F} : L^1(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$

- When $p = 2$, $\mathcal{F}f$ is in $L^2(\mathbb{R}^d)$ so it is arbitrary on any zero measure subset

- The Fourier transform of a L^p function, $p > 1$ cannot be always restricted to hyperplanes as shown by the following example

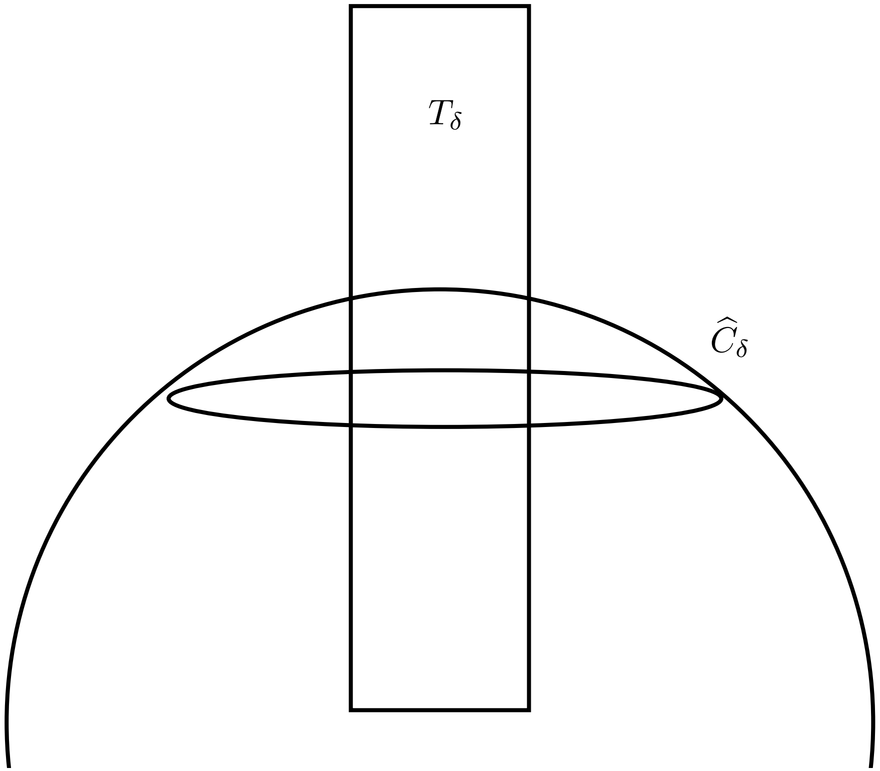
$$f(x) = \frac{e^{-|x'|}}{1 + |x_1|}$$

whose Fourier transform cannot be restricted to

$$\{\xi \in \mathbb{R}^d : \xi_1 = 0\}$$

- The index $p_0 = \frac{2(d+1)}{d+3} = 2 - \frac{4}{d+3}$ in Tomas-Stein's theorem is optimal according to Knapp's counter-example which is given by g_δ $(\delta > 0)$ the characteristic function of the spherical cap

$$\widehat{C}_\delta = \{x \in S : |x \cdot e_d| < \delta\}$$



Obviously $|\widehat{C}_\delta| \sim \delta^{d-1} \Rightarrow \|g_\delta\|_{L^2(S^{d-1})} \sim \delta^{(d-1)/2}$

Second consider T_δ the tube in the x space oriented orthogonally to the sphere so that

$$|x'| \leq \delta^{-1}, \qquad |x_d| \leq \delta^{-2}$$

Then for x in T_δ, the quantity $x \cdot \xi$ is almost zero for $\xi \in \widehat{C}_\delta$ which implies that

$$|\widehat{g_\delta \sigma}(x)| = \left| \int_{\widehat{C}_\delta} e^{ix\cdot\xi} d\sigma(\xi) \right| \sim |\widehat{C}_\delta| \sim \delta^{d-1}$$

Therefore (since $|T_\delta| \sim \delta^{-d-1}$)

$$\|\widehat{g_\delta \sigma}\|_{L^{p'}(\mathbb{R}^d)} \geq \left(\int_{T_\delta} |\widehat{g\sigma}(x)|^{p'} dx \right)^{1/p'}$$
$$\sim \left(\int_{T_\delta} \delta^{(d-1)p'} dx \right)^{1/p'} \sim \delta^{(d-1)} \delta^{-(d+1)/p'}$$

Recalling that the (dual) restriction estimate is

$$\|\mathcal{F}^{-1}(gd\sigma)\|_{L^{p'}(\mathbb{R}^d)} \leq C\|g\|_{L^2(S,d\sigma)}$$

It gives for g_δ

$$\delta^{(d-1)}\delta^{-(d+1)/p'} \lesssim \delta^{(d-1)/2}$$

which is valid only if

$$d-1-\frac{d+1}{p'} \geq \frac{d-1}{2} \Rightarrow p \leq p_0$$

This shows that the sharpness of the index

$$p_0 = \frac{2(d+1)}{d+3} = 2 - \frac{4}{d+3}$$

in Tomas-Stein's theorem

Note that Knapp's counter-example involves in the study of maximizers for Tomas-Stein's inequality : Christ-Shao, Frank-Lieb-Sabin, Shao,...

- There are also some restriction results (with smaller range of possible exponents) when the hypothesis of curvature is weaker

Buschenhenke-Müller-Vargas, Hickman-Rogers, Ikromov-Müller, Müller-Ricci-Wright, Tao, Wang,...

- For the Heisenberg group $\mathbb{H}^d$ (which is a step 2 stratified Lie group) that can be identified to $\mathbb{R}^{2d} \times \mathbb{R}$, we have the following Fourier restriction theorem due to Müller (where $(Y, s) \in \mathbb{R}^{2d} \times \mathbb{R}$ is a generic element of $\mathbb{H}^d$), $1 \leq p \leq 2$

$$\|\mathcal{F}_{\mathbb{H}}(f)_{|\mathbb{S}_{\widehat{\mathbb{H}}^d}}\|_{L^2(\mathbb{S}_{\widehat{\mathbb{H}}^d})} \leq C\|f\|_{L^p_Y L^1_s}$$

To be aware : contrary to $\mathbb{R}^d$ the Fourier dual of $\mathbb{H}^d$ is not $\mathbb{H}^d$!

No gain in s direction, but the gain is better than in $\mathbb{R}^{2d}$ for Y

Restriction theory has many applications to other topics, from number theory to PDEs :

Let us see how it provides Strichartz estimates for instance for the Schrödinger equation. We have seen that :

$$u(t,x) = (2\pi)^{-d} \int e^{i(x\cdot\xi + t|\xi|^2)} \mathcal{F}(u_0)(\xi) d\xi.$$

This formula can be interpreted as the restriction of the Fourier transform on the paraboloid S in the space of frequencies of $\mathbb{R}^{1+d}$, defined as

$$S = \left\{ (\alpha, \xi) \in \mathbb{R}^{1+d} \mid \alpha = |\xi|^2 \right\}.$$

Then (with $y = (t,x)$ and $z = (\alpha, \xi)$)

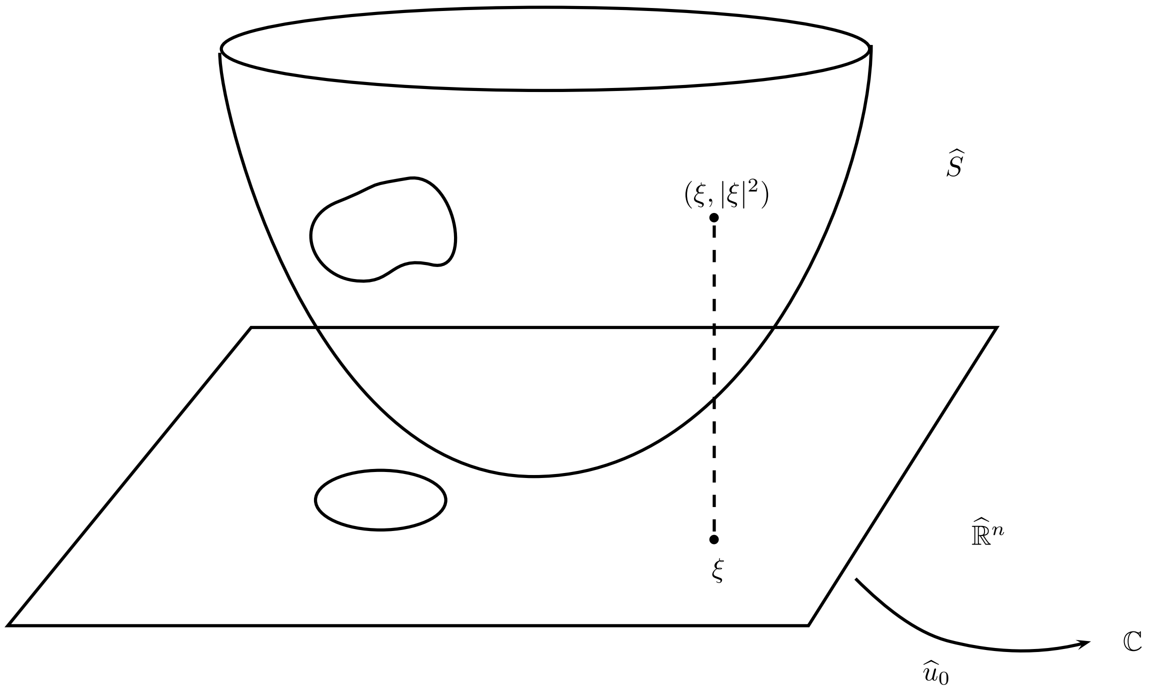
$$u(t,x) = (2\pi)^{-d} \int_S e^{iy\cdot z} g(z) d\sigma(z),$$

where $g(|\xi|^2, \xi) = \mathcal{F}(u_0)(\xi)$

Invoking the dual form of Tomas-Stein theorem, we deduce that

$$\|u\|_{L^{2+\frac{4}{d}}(\mathbb{R}, L^{2+\frac{4}{d}}(\mathbb{R}^d))} \leq C \|u_0\|_{L^2(\mathbb{R}^d)}$$

since $\|g\|_{L^2(S,d\sigma)} = \|u_0\|_{L^2(\mathbb{R}^d)}$



This proof is due to Strichartz

Kato smoothing effect

- This property was first established by Kato for KdV based on Fourier restriction type results, then by Constantin-Saut for systems including the Schrödinger equation (see also Ben Artzi-Devinatz, Ben Artzi-Klainerman, Sjölin, Vega, Yajima,...)

- In the case of the Schrödinger equation, even though $\|u(t,\cdot)\|_{L^2} = \|u_0\|_{L^2}$ $(\mathcal{F}u(t,\xi) = e^{it|\xi|^2}\mathcal{F}(u_0)(\xi))$ one can locally gain one half derivative in the following sense (book of Robbiano : Smoothing effects for the Schrödinger equation)

There exists a positive constant C such that, for all $u_0 \in L^2(\mathbb{R}^d)$, the solution u of the Schrödinger equation satisfies :

$$\|\langle x\rangle^{-1}\langle D_x\rangle^{\frac{1}{2}}u\|_{L^2(\mathbb{R}\times\mathbb{R}^d)} \leq C\|u_0\|_{L^2(\mathbb{R}^d)}\,.$$

The gain of 1/2-derivative is optimal Sun-Trélat-Zhang-Zhong

Framework

In these lectures, I will limit myself to the study of dispersion phenomena on $\mathbb{R}^d$ and on stratified Lie groups, mainly on the Heisenberg group $\mathbb{H}^d$ which is the most renowned example of step 2 stratified Lie groups and on the Engel group which is a step 3 stratified Lie group

- Recall that one can recover the group from its Lie algebra and vice versa, by means of the formula of Baker-Campbell-Hausdorff and the exponential map

- The Lie algebra $\mathfrak{g}$ of a step-r stratified Lie group admits the following stratification

$$\mathfrak{g} = \oplus_{i=1}^{r} \mathfrak{g}_i \quad \text{with} \quad \mathfrak{g}_{i+1} = [\mathfrak{g}_1, \mathfrak{g}_i]$$

where $\mathfrak{g}_1$ is Lie bracket generating. The most famous examples are

Heisenberg group (step-2) $\left(\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2, \overbrace{X_1, X_2}^{\mathfrak{g}_1}, \overbrace{[X_1, X_2]}^{\mathfrak{g}_2}\right)$

Engel group G (step-3) $\left(\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3, \overbrace{X_1, X_2}^{\mathfrak{g}_1}, \overbrace{X_3 = [X_1, X_2]}^{\mathfrak{g}_2}, \overbrace{[X_1, X_3]}^{\mathfrak{g}_3}\right)$

As we shall see, the study of dispersive phenomena on stratified Lie groups is a challenging task : contrary to $\mathbb{R}^d$, the solution is not one oscillating integral, but rather a series of oscillating integrals

- As mentioned above, for the free linear Schrödinger equation on $\mathbb{R}^d$,

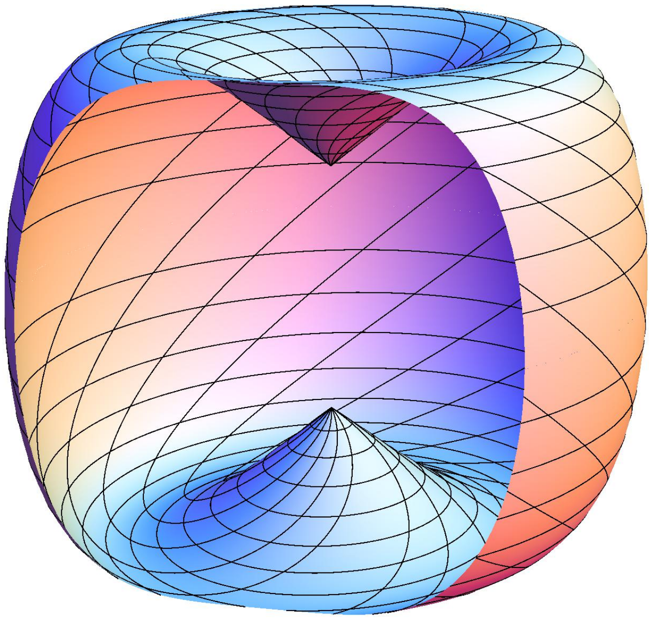
$$u(t,\cdot) = \frac{e^{i\frac{|\cdot|^2}{4t}}}{(4\pi i t)^{\frac{d}{2}}} \star u_0$$

and then we have dispersive estimates with an optimal rate of decay $r = d/2$ (the optimality follows from the stationary phase theorem)

- However on $\mathbb{H}^d$, the Schrödinger equation is a totally non dispersive equation : it behaves as a transport equation along the center which is generated by the variable s Bahouri-Gérard-Xu

The behavior of the Schrödinger equation is one of the remarkable differences between $\mathbb{R}^d$ and $\mathbb{H}^d$, but as we shall see there are many similarities : Haar measure, distance, derivations,...

Below the unit sphere on the Heisenberg group



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Stein-Weiss, Introduction to Fourier analysis on Euclidean spaces, Princeton Mathematical Series (Chapter IV)

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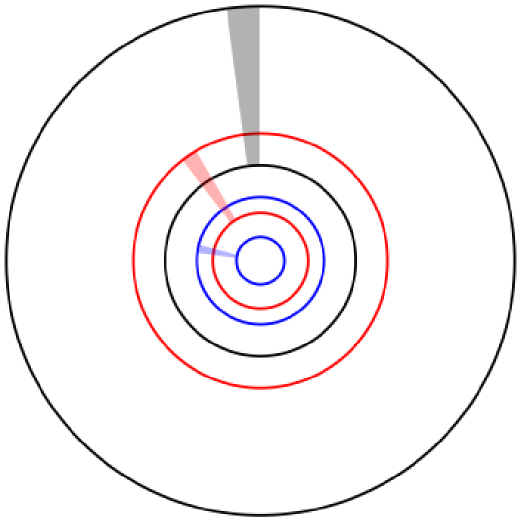
$$u = \sum_{\text{support}(\mathcal{F}\Delta_j u) \subset 2^j \mathcal{C}} \underbrace{\Delta_j u}$$



i) Bernstein inequalities : two fundamental inequalities

$$\text{Supp}\mathcal{F}u \subset \lambda\mathcal{C} \Rightarrow \|D^\alpha u\|_{L^p(\mathbb{R}^d)} \sim \lambda^{|\alpha|} \|u\|_{L^p(\mathbb{R}^d)}$$

$$\text{Supp}\mathcal{F}u \subset \lambda\mathcal{B} \text{ and } q \geq p \Rightarrow \|u\|_{L^q(\mathbb{R}^d)} \leq C\lambda^{d(1/p-1/q)} \|u\|_{L^p(\mathbb{R}^d)}$$

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