Exercise 1

1. For all $t \in \mathbb{R}$ we have $(g \circ f)^{-1}((t, \infty]) = f^{-1}(g^{-1}((t, \infty])) \in \mathcal{A}$.

2. The map $h' : \mathbb{R}^2 \to \mathbb{R}$, $(a, b) \mapsto a+b$ is measurable since the preimage of an open set is an open set, and the map $h : X \to \mathbb{R}^2$, $x \mapsto (f(x), g(x))$ is measurable since $h^{-1}(A_1 \times A_2) = f^{-1}(A_1) \cap g^{-1}(A_2) \in \mathcal{A}$ for any $A_1, A_2 \subset \mathbb{R}$ measurable. Hence, $f+g = h' \circ h$ is measurable.

3. For $g_1(x) = \inf_{n\in\mathbb{N}} f_n(x)$ and $g_2(x) = \sup_{n\in\mathbb{N}} f_n(x)$ and $t \in \mathbb{R}$ we have $g_1^{-1}([-\infty, t)) = \cup_{n\in\mathbb{N}}\{f_n < t\}$, $g_2^{-1}((t, \infty]) = \cup_{n\in\mathbb{N}}\{f_n > t\}$ which is measurable. Therefore $\liminf_{n\to\infty} f_n = \sup_{n\in\mathbb{N}} \inf_{k\geq n} f_k$ and $\limsup_{n\to\infty} f_n = \inf_{n\in\mathbb{N}} \sup_{k\geq n} f_k$ are measurable which implies that $\lim_{n\to\infty} f_n$ is measurable.

4. We set $A_k^n = f^{-1}([2^{-n}(k-1), 2^{-n}k))$ for $k = 1, ..., n2^n$, $n \in \mathbb{N}_0$, and $f_n = \sum_{k=1}^{n2^n} 2^{-n}(k-1)\chi_{A_k^n} + \infty \cdot \chi_{\{f=\infty\}}$. Then $(f_n)_{n\in\mathbb{N}}$ has the desired properties.

5. We first compute that for $f : X \to [0, \infty]$, $A \in \mathcal{A}$ and $a \in \mathbb{R}$ we have $\int_X f + a\chi_A d\mu = \int_X f d\mu + a\mu(A)$ which yields the claim if f and g are simple functions. If $f, g : X \to [0, \infty]$ we take simple functions $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ with the properties from 4. to infer that $\int_X f + g d\mu = \lim_{n\to\infty} \int_X f_n + g_n d\mu = \lim_{n\to\infty} \int_X f_n d\mu + \lim_{n\to\infty} \int_X g_n d\mu = \int_X f d\mu + \int_X g d\mu$. Note that we can apply the same argument in the case that $f \geq g \geq 0$ almost everywhere on X and take the sequences provided by 4. to obtain $\int_X f - g d\mu = \int_X f d\mu - \int_X g d\mu$. If $f, g : X \to \mathbb{R} \cup \{\pm\infty\}$ we set $E^+ = \{f + g \geq 0\}$, $E^- = \{f + g < 0\}$ and noticing that $(f+g)\chi_{E^+} = (f^+ + g^+)\chi_{E^+} - (f^- + g^-)\chi_{E^+}$, $(f+g)\chi_{E^-} = (f^- + g^-)\chi_{E^-} - (f^+ + g^+)\chi_{E^-}$ we compute $\int_X f + g d\mu = \int_X (f + g)\chi_{E^+} d\mu - \int_X (f + g)\chi_{E^-} d\mu = \int_X f^+ d\mu - \int_X f^- d\mu + \int_X g^+ d\mu - \int_X g^- d\mu = \int_X f d\mu + \int_X g d\mu$. If $f, g : X \to \mathbb{C} \cup \{\pm\infty\}$ we consider $\mathrm{Re}(f + g)$ and $\mathrm{Im}(f + g)$.

Exercise 2

1. By Hölder's inequality j is well-defined, $\|j(g)\|_{(L^p)^*} \leq \|g\|_{L^q}$, and $\|g\|_{L^q}^q = j(g)(g|g|^{q-2}) \leq \|j(g)\|_{(L^p)^*}\|g\|_{L^q}^{q/p}$ which implies that j is an isometry, in particular it is injective. If $\varphi \in (L^p)^*$, $\varphi \neq 0$ we set $N = \{f \in L^p : \varphi(f) = 0\}$ which is a closed subspace. We take $f_1 \in L^p$ such that $\varphi(f_1) \neq 0$ and set $f_0 = (\varphi(f_1))^{-1}f_1$ which satisfies $\varphi(f_0) = 1$. Let $g_0 = f_0 - p(f_0)$, where $p : L^p \to N$ is the projection from Lemma 3.22. Then $\varphi(g_0) = 1$ and since $\mathrm{Re} \int_X (f - p(f_0))\overline{g_0}|g_0|^{p-2} d\mu \leq 0$ for all $f \in N$ and N is a subspace it follows that $\int_X f\overline{g_0}|g_0|^{p-2} d\mu = 0$ for all $f \in N$. This together with the fact that $f - \varphi(f)g_0 \in N$ for all $f \in L^p$ implies that $\int_X f\overline{g_0}|g_0|^{p-2} d\mu = \int_X \varphi(f)g_0\overline{g_0}|g_0|^{p-2} d\mu = \varphi(f)\|g_0\|_{L^p}^p$ and hence $\varphi(f) = j(g_0|g_0|^{p-2}\|g_0\|_{L^p}^{-p})(f)$ for all $f \in L^p$.

2. j is antilinear and well-defined by Hölder's inequality which also yields $\|j(g)\|_{(L^1)^*} \leq \|g\|_{L^\infty}$. For $\epsilon > 0$ we define $E_\epsilon = \{g > \|g\|_{L^\infty} - \epsilon\}$ and let $(X_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\mu(X_n) < \infty$, $X_n \subset X_{n+1}$, $X = \cup_{n\in\mathbb{N}} X_n$. Then $j(g)(\chi_{E_\epsilon \cap X_n}) \geq (\|g\|_{L^\infty} - \epsilon)\|\chi_{E_\epsilon \cap X_n}\|_{L^1}$ which implies $\|j(g)\|_{(L^1)^*} \geq \|g\|_{L^\infty}$. Therefore j is an isometry and hence injective. Let $\varphi \in (L^1)^*$. Note that for any $A \in \mathcal{A}$, $\mu(A) < \infty$, and $f \in L^p$, $1 < p < \infty$, we have

$f\chi_A \in L^1$ since $\|f\chi_A\|_{L^1} \leq \|f\|_{L^p}\|\chi_A\|_{L^{\frac{p}{p-1}}}$ by Hölder's inequality, where $\chi_A \in L^{\frac{p}{p-1}}$ since $\mu(A) < \infty$. Therefore if we define $\varphi_A(f) := \varphi(f\chi_A)$ then $\varphi_A \in (L^p)^*$ for any $1 < p < \infty$. By 1. there exists $g_{A,p} \in L^{\frac{p}{p-1}}$ such that for any $f \in L^p$, $\varphi_A(f) = \int_X fg_{A,p}d\mu$ for any $1 < p < \infty$, $A \in \mathcal{A}$ such that $\mu(A) < \infty$. We claim that $g_{A,p} = g_{A,\tilde{p}}$ for all $p, \tilde{p} \in (1,2)$, $A \in \mathcal{A}$ such that $\mu(A) < \infty$. Without loss of generality we assume that $p < \tilde{p}$. For $f \in L^p$, $\tilde{f} \in L^{\tilde{p}}$ we have $\varphi(\chi_A(f - \tilde{f})) = \varphi(\chi_A f) - \varphi(\chi_A \tilde{f}) = \int_X fg_{A,p}d\mu - \int_X \tilde{f}g_{A,\tilde{p}}d\mu = \int_X fg_{A,p} - \tilde{f}g_{A,\tilde{p}}d\mu$. By Hölder's inequality for any $n \in \mathbb{N}$ it holds that $\tilde{f}\chi_{X_n} \in L^p$. Thus choosing $f = \tilde{f}\chi_{X_n}$ and n sufficiently large such that $A \subset X_n$ yields $0 = \varphi(\chi_A(\tilde{f}\chi_{X_n} - \tilde{f})) = \int_X \tilde{f}(\chi_{X_n}g_{A,p} - g_{A,\tilde{p}})d\mu$ for any $\tilde{f} \in L^{\tilde{p}}$. Now we choose $\tilde{f} = (\overline{\chi_{X_n}g_{A,p} - g_{A,\tilde{p}}})\chi_{X_n}$. Note that then $\tilde{f} \in L^{\tilde{p}}$ for $\tilde{p} \in (1,2)$ by Hölder's inequality. We obtain $0 = \int_X \chi_{X_n}|\chi_{X_n}g_{A,p} - g_{A,\tilde{p}}|^2 d\mu$ for any $n \in \mathbb{N}$ sufficiently large, from which it follows that $g_{A,p} = g_{A,\tilde{p}}$ almost everywhere for any $p, \tilde{p} \in (1,2)$. Thus $g_A := g_{A,p}$, $p \in (1,2)$, $A \in \mathcal{A}$, $\mu(A) < \infty$, is well-defined. Next we show that if $A, B \in \mathcal{A}$, $B \subset A$, then $g_A = g_B$ almost everywhere on B. We have $\varphi(\chi_A f) = \int_X fg_A d\mu$, $\varphi(\chi_B f) = \int_X fg_B d\mu$ for all $f \in L^p$, $p \in (1,2)$. Choosing $f = \chi_{A\cap B}(\overline{g_A - g_B})$ (which is in L^p for $p \in (1,2)$) we obtain $0 = \int_X \chi_{A\cap B}|g_A - g_B|^2 d\mu$ which yields $g_A = g_B$ almost everywhere on $A \cap B = B$. Therefore if we define $g(x) = g_{X_n}(x)$ if $x \in X_n$ for almost every $x \in X$, then g is well-defined and satisfies $\varphi(\chi_A) = \int_A g d\mu$ for all $A \in \mathcal{A}$, $\mu(A) < \infty$. By the linearity of the integral and φ we also obtain $\varphi(f) = \int_X fg d\mu$ for all simple functions $f = \sum_{i=1}^m a_i\chi_{A_i}$ such that $\mu(A_i) < \infty$. We claim that $g \in L^\infty$. Assume otherwise, then for every $n \in \mathbb{N}$, $\mu(\{|g| > n\}) > 0$. Let $A \in \mathcal{A}$, $0 < \mu(A) < \infty$ (which exists since X is σ finite), and set $f_n = \chi_{A\cap\{|g|>n\}} \in L^1$. Then $\|\varphi\|_{(L^1)^*}\|f_n\|_{L^1} \geq \varphi(f_n) = \int_X f_n g d\mu \geq n\mu(A\cap\{|g| > n\})$ and since $\|f_n\|_{L^1} = \mu(A \cap \{|g| > n\})$ this is a contradiction to $\varphi \in (L^1)^*$. Next let $f \in L^1$ and let $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ be sequences of simple functions for f^+, f^- as in Exercise 1, 4. We know that for all $n \in \mathbb{N}$, $\varphi(f_n) = \int_X f_n g d\mu$, $\varphi(g_n) = \int_X g_n g d\mu$, and since φ is continuous we have $\varphi(f_n) \to \varphi(f^+)$, $\varphi(g_n) \to \varphi(f^-)$. Moreover, almost everywhere on X we have $|f_n g| \leq f^+\|g\|_{L^\infty}$, $|g_n g| \leq f^-\|g\|_{L^\infty}$ so we can apply the dominated convergence theorem to deduce $\varphi(f^+) = \int_X f^+ g d\mu$, $\varphi(f^-) = \int_X f^- g d\mu$ and hence $\varphi(f) = \int_X fg d\mu$.