Exercise 1

Define $T : l^1(\mathbb{N}) \to (c_0)^*$, $Ty(x) = \sum_{j=1}^\infty x_j y_j$. By Hölder's inequality we have $\|Ty\|_{(c_0)^*} \le \|y\|_{l^1}$ for all $y \in l^1(\mathbb{N})$. Moreover, we define $S : (c_0)^* \to l^1(\mathbb{N})$, $S\varphi = (\varphi(e_j))_{j\in\mathbb{N}}$, where $(e_j)_k = \delta_{jk}$. Note that S is well-defined since for all $x \in c_0$, $\sum_{j=1}^\infty x_j\varphi(e_j) = \lim_{J\to\infty} \varphi(\sum_{j=1}^J x_j e_j) = \varphi(x)$, from which we deduce that $\sum_{j=1}^\infty x_j\varphi(e_j)$ converges. The Banach-Steinhaus theorem yields $S\varphi \in l^1(\mathbb{N})$ with $\|S\varphi\|_{l^1} \le \sup_{n\in\mathbb{N}} \sup_{x\in c_0, \|x\|_{l^\infty}\le 1} |\sum_{j=1}^n x_j\varphi(e_j)| \le \|\varphi\|_{(c_0)^*}$. Note that $ST = Id_{l^1(\mathbb{N})}$, $TS = Id_{(c_0)^*}$, which implies that T is a linear isomorphism. The inequalities $\|Ty\|_{(c_0)^*} \le \|y\|_{l^1}$ and $\|S\varphi\|_{l^1} \le \|\varphi\|_{(c_0)^*}$ for all $y \in l^1(\mathbb{N})$, $\varphi \in (c_0)^*$ imply that T is an isometry.

Exercise 2

1. "$\Rightarrow$" The boundedness is clear. If for $n \in \mathbb{N}$, e_n denotes the sequence defined by $(e_n)_j = \delta_{jn}$ ($j \in \mathbb{N}$), then $x_n^{(k)} = \sum_{j=1}^\infty (e_n)_j x_j^{(k)} \to \sum_{j=1}^\infty (e_n)_j x_j = x_n$ as $k \to \infty$ for all $n \in \mathbb{N}$.
   "$\Leftarrow$" Let $y \in l^{\frac{p}{p-1}}(\mathbb{N})$, $\epsilon > 0$. There exist $J_0, k_0 \in \mathbb{N}$ such that $(\sum_{j=J_0+1}^\infty |y_j|^{\frac{p}{p-1}})^{\frac{p-1}{p}} < \epsilon$, and $|x_j^{(k)} - x_j| < \frac{\epsilon}{J_0+1}$ for all $j = 1, ..., J_0$, $k \ge k_0$. Then we have for $k \ge k_0$, $|\sum_{j=1}^\infty x_j^{(k)} y_j - \sum_{j=1}^\infty x_j y_j| \le (\sup_{j=1,...,J_0} |y_j|) \sum_{j=1}^{J_0} |x_j^{(k)} - x_j| + \|x^{(k)} - x\|_{l^p} (\sum_{j=J_0+1}^\infty |y_j|^{\frac{p}{p-1}})^{\frac{p-1}{p}} \le C\epsilon$ for some constant $C > 0$ independent of ϵ and k.

2. The implication from right to left is clear. Let $(x^{(k)})_{k\in\mathbb{N}} \subset l^1(\mathbb{N})$, $x \in l^1(\mathbb{N})$ such that $x^{(k)} \rightharpoonup x$ in $l^1(\mathbb{N})$ as $k \to \infty$, and assume there exists $\epsilon > 0$ such that $\limsup_{k\to\infty} \sum_{j=1}^\infty |x_j^{(k)} - x_j| > \epsilon$. Let $k_1 \in \mathbb{N}$ be minimal such that $\sum_{j=1}^\infty |x_j^{(k_1)} - x_j| > \epsilon$, $J_1 \in \mathbb{N}$ minimal such that $\sum_{j=1}^{J_1} |x_j^{(k_1)} - x_j| > \frac{\epsilon}{2}$ and $\sum_{j=J_1+1}^\infty |x_j^{(k_1)} - x_j| < \frac{\epsilon}{5}$. For $n \in \mathbb{N}$, $n \ge 2$, let $k_n \in \mathbb{N}$, $k_n \ge k_{n-1}$, be minimal such that $\sum_{j=1}^\infty |x_j^{(k_n)} - x_j| > \epsilon$ and $\sum_{j=1}^{J_{n-1}} |x_j^{(k_n)} - x_j| < \frac{\epsilon}{5}$, and let $J_n \in \mathbb{N}$, $J_n \ge J_{n-1}$, be minimal such that $\sum_{j=J_{n-1}+1}^{J_n} |x_j^{(k_n)} - x_j| > \frac{\epsilon}{2}$ and $\sum_{j=J_n+1}^\infty |x_j^{(k_n)} - x_j| < \frac{\epsilon}{5}$. For $j \in \mathbb{N}$, let $c_j = \mathrm{sgn}(x_j^{(k_1)} - x_j)$, if $1 \le j \le J_1$, $c_j = \mathrm{sgn}(x_j^{(k_{n+1})} - x_j)$, if $J_n < j \le J_{n+1}$, for $n \in \mathbb{N}$. Then $(c_j)_{j\in\mathbb{N}} \in l^\infty$ and for all $n \in \mathbb{N}$, $|\sum_{j=1}^\infty c_j(x_j^{(k_n)} - x_j)| \ge \sum_{j=J_{n-1}+1}^{J_n} |x_j^{(k_n)} - x_j| - \sum_{j=1}^{J_{n-1}} |x_j^{(k_n)} - x_j| - \sum_{j=J_n+1}^\infty |x_j^{(k_n)} - x_j| \ge \frac{\epsilon}{10}$, which is a contradiction.

Exercise 3

1. Let $x, y \in X$, $\alpha > 0$. We have $p_C(\alpha x) = \inf\{\alpha\lambda > 0 : \frac{1}{\lambda}x \in C\} = \alpha p_C(x)$. Note that the equality also holds if $p_C(x) = \infty$ or $p_C(\alpha x) = \infty$. If $p_C(x) = \infty$ or $p_C(y) = \infty$, then $p_C(x + y) \le p_C(x) + p_C(y)$ is clear. Let $p_C(x) < \infty$ and $p_C(y) < \infty$. If $\lambda_x, \lambda_y > 0$ such that $\frac{1}{\lambda_x}x, \frac{1}{\lambda_y}y \in C$, then $\frac{x+y}{\lambda_x+\lambda_y} = (\frac{\lambda_x}{\lambda_x+\lambda_y})\frac{1}{\lambda_x}x + (\frac{\lambda_y}{\lambda_x+\lambda_y})\frac{1}{\lambda_y}y \in C$. Taking the infimum over λ_x and λ_y yields $p_C(x + y) \le p_C(x) + p_C(y)$.

2. Since for all $x \in X$ there exists $\lambda > 0$ such that $\frac{1}{\lambda}x \in C$, we have $\{\lambda > 0 : \frac{1}{\lambda}x \in C\} \ne \emptyset$, and thus $p_C(x) < \infty$.

3. For $x \in X$ we can write $p_{B_1(0)}(x) = \inf\{\lambda > 0 : \frac{\|x\|}{\lambda} = 1\} = \|x\|$.

Exercise 4

It suffices to show the claim for $\mathbb{K} = \mathbb{R}$. For $x_0 \in C$ let $U = C - x_0 = \{y - x_0 : y \in C\}$. Then U is convex, open and $0 \in U$. We set $y_0 = -x_0$, $Y = \operatorname{span}\{y_0\}$, and define $l(ty_0) = tp_U(y_0)$ for $t \in \mathbb{R}$. Then $l \leq p_U$ on Y, and by Hahn-Banach there exists $x^* \in X^*$ such that $x^*|_Y = l$ and $x^* \leq p_U$ on X. Note that $x^*(y_0) = l(y_0) = p_U(y_0) \geq 1$ and $x^*(x + y_0) \leq p_U(x + y_0) < 1$ for all $x \in C$. This implies $x^*(x) = x^*(x + y_0) - x^*(y_0) < 0$ for all $x \in C$.

Exercise 5

Let $p \in (1, 2)$ and recall Hanner's inequality $\|f + g\|_{L^p}^p + \|f - g\|_{L^p}^p \geq (\|f\|_{L^p} + \|g\|_{L^p})^p + |\|f\|_{L^p} - \|g\|_{L^p}|^p$ for $f, g \in L^p(U)$. By the definition of $\|\cdot\|_{W^{k,p}(U)}$, it suffices to prove the claim for $L^p(U)$. Let $f, g \in L^p(U)$. For $(u, v) \in [0, \infty)^2$ define $\xi(u, v) = (u + v)^p + |u - v|^p$. Then ξ is symmetric and $\xi(u, \cdot)$ and $\xi(\cdot, v)$ are strictly increasing for all $u, v \in [0, \infty)$. We can write the above Hanner's inequality as $\|f + g\|_{L^p}^p + \|f - g\|_{L^p}^p \geq \xi(\|f\|_{L^p}, \|g\|_{L^p})$. Let $\epsilon \in (0, 2)$. Let $\|f\|_{L^p}$, $\|g\|_{L^p} \leq 1$ with $\|f - g\|_{L^p} \geq \epsilon$, and set $h_1 = \frac{1}{2}(f + g)$, $h_2 = \frac{1}{2}(f - g)$. Then $2 \geq \xi(\|h_1\|_{L^p}, \|h_2\|_{L^p}) \geq \xi(\|h_1\|_{L^p}, \frac{\epsilon}{2})$. Due to $\xi(1, \frac{\epsilon}{2}) > 2$, $\xi(0, \frac{\epsilon}{2}) < 2$, there exists a unique $\delta > 0$ such that $\xi(1 - \delta, \frac{\epsilon}{2}) = 2$. It follows that $\|h_1\|_{L^p} \leq 1 - \delta$, which implies $\delta(\epsilon) \geq \delta$. Let $2 \leq p < \infty$, $f, g \in L^p(U)$, $\|f\|_{L^p}$, $\|g\|_{L^p} \leq 1$ with $\|f - g\|_{L^p} \geq \epsilon$. In the proof of Hanner's inequality we have seen that $\alpha(r)\|f\|_{L^p}^p + \beta(r)\|g\|_{L^p}^p \geq \|f + g\|_{L^p}^p + \|f - g\|_{L^p}^p$, for $r \in [0, 1]$, where $\alpha(r) = (1 + r)^{p-1} + (1 - r)^{p-1}$, $\beta(r) = ((1 + r)^{p-1} - (1 - r)^{p-1})r^{1-p}$. For $r = 1$ we obtain $\|f + g\|_{L^p}^p + \|f - g\|_{L^p}^p \leq 2^{p-1}(\|f\|_{L^p}^p + \|g\|_{L^p}^p)$, and thus $\|h_1\|_{L^p}^p + \|h_2\|_{L^p}^p \leq \frac{\|f\|_{L^p}^p}{2} + \frac{\|g\|_{L^p}}{2} \leq 2$, so that we arrive at $1 - \|\frac{1}{2}(f + g)\|_{L^p}^p \geq (\frac{\epsilon}{2})^p$. It follows that for some $c > 0$, $1 - \frac{1}{2}\|f + g\|_{L^p} \geq c^{-1}(1 - \|\frac{1}{2}(f + g)\|_{L^p}^p)^{1/p} \geq c^{-1}\frac{\epsilon}{2}$.