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Solution to Problem Sheet 6

Bifurcation Theory

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Problem 16:
Determine all bifurcation points $(0, \lambda_0) \in \mathbb{R}^2 \times \mathbb{R}$ of the nonlinear system

$$\begin{cases} \sin(x_1 + \lambda x_2) = x_1, \\ \cos(\lambda x_1 + x_2) = 1 + x_1. \end{cases} \tag{1}$$

Solution to problem 16:
Claim: $(0, \lambda_0)$ is a bifurcation point of problem (1) if and only if $\lambda_0 = 0$.

Proof: Solving problem (1) is equivalent to finding zeros of the function

$$F : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2, \quad F(x_1, x_2, \lambda) := \begin{pmatrix} \sin(x_1 + \lambda x_2) - x_1 \\ \cos(\lambda x_1 + x_2) - 1 - x_1 \end{pmatrix}.$$

Then $F(0, 0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$, and F is twice continuously differentiable with

$$F_x(x_1, x_2, \lambda) = \begin{pmatrix} \cos(x_1 + \lambda x_2) - 1 & \lambda \cos(x_1 + \lambda x_2) \\ -\lambda \sin(\lambda x_1 + x_2) - 1 & -\sin(\lambda x_1 + x_2) \end{pmatrix},$$

$$F_{x\lambda}(x_1, x_2, \lambda) = \begin{pmatrix} -x_2 \sin(x_1 + \lambda x_2) & \cos(x_1 + \lambda x_2) - \lambda x_2 \sin(x_1 + \lambda x_2) \\ -\sin(\lambda x_1 + x_2) - \lambda x_1 \cos(\lambda x_1 + x_2) & -x_1 \cos(\lambda x_1 + x_2) \end{pmatrix}$$

for $x_1, x_2, \lambda \in \mathbb{R}$, and in particular, for $\lambda_0 \in \mathbb{R}$,

$$F_x(0, 0, \lambda_0) = \begin{pmatrix} 0 & \lambda_0 \\ -1 & 0 \end{pmatrix}, \qquad F_{x\lambda}(0, 0, \lambda_0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

If $\lambda_0 \neq 0$, we have that $\det F_x(0, 0, \lambda_0) = \lambda_0 \neq 0$; hence $F_x(0, 0, \lambda_0)$ (interpreted as a linear mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$) is a homeomorphism. Proposition 3.9 then states that $(0, \lambda_0)$ cannot be a bifurcation point for problem (1).

Now let $\lambda_0 = 0$ and apply the Crandall-Rabinowitz Bifurcation Theorem: We have

$$\ker F_x(0, 0, 0) = \operatorname{span} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \qquad \operatorname{ran} F_x(0, 0, 0) = \operatorname{span} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

and thus, the simplicity condition (S) holds. Moreover, with $\varphi := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we have

$$F_{x\lambda}(0, 0, 0)[\varphi] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \notin \operatorname{ran} F_x(0, 0, 0),$$

which shows that the transversality condition (T) is also satisfied. Theorem 4.3 now states that $(0, 0, 0)$ is a bifurcation point. □

Problem 17:
Let $A \in \mathbb{R}^{n\times n}$ be symmetric, and $J \in \mathbb{R}^{n\times n}$. For $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we study the equation

$$Ax = \lambda x + |x|^2 Jx. \tag{2}$$

Discuss the existence of nontrivial solutions in a neighborhood of $(0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}$ if ...

(a) λ_0 is not an eigenvalue of A,

(b) λ_0 is a simple eigenvalue of A.

Solution to problem 17:
Solving problem (2) is equivalent to finding zeros of the function

$$F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad F(x, \lambda) := Ax - \lambda x - |x|^2 Jx.$$

We note that $F(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$, and that F is twice continuously differentiable with

$$F_x(x, \lambda)[h] = Ah - \lambda h - |x|^2 Jh - 2\langle x|h\rangle Jx, \qquad F_{x\lambda}(x, \lambda)[h] = -h$$

for $x, h, \lambda \in \mathbb{R}$; moreover, for $\lambda_0 \in \mathbb{R}$,

$$F_x(0, \lambda)[h] = (A - \lambda I)h, \qquad F_{x\lambda}(0, \lambda)[h] = -h.$$

(a) Let λ_0 be not an eigenvalue of A. *Claim:* There are no nontrivial solutions of problem (2) in a neighborhood of $(0, \lambda_0)$.

Proof: This is a consequence of Proposition 3.9, as $F_x(0, \lambda_0) = A - \lambda_0 I : \mathbb{R}^n \to \mathbb{R}^n$ is invertible. □

(b) Let λ_0 be a simple eigenvalue of A. *Claim:* Nontrivial solutions of problem (2) form a continuous branch bifurcating from $(0, \lambda_0)$.

Proof: We check the assumptions of the Crandall-Rabinowitz Bifurcation Theorem. Since A is assumed symmetric, there exist eigenpairs $(\psi_j, \mu_j) \in \mathbb{R}^n \times \mathbb{R}$ with $A\psi_j = \mu_j \psi_j$ and $\{\psi_1, ..., \psi_n\}$ being a complete orthonormal subset of $\mathbb{R}^n$.

Without loss of generality, $\mu_1 = \lambda_0$ and by assumption (simple eigenvalue), we have $\mu_j \neq \lambda_0$ for $j > 1$. Thus,

$$\ker F_x(0, \lambda_0) = \ker(A - \lambda_0 I) = \operatorname{span}\{\psi_1\},$$
$$\operatorname{ran} F_x(0, \lambda_0) = \operatorname{ran}(A - \lambda_0 I) = \operatorname{span}\{\psi_2, ..., \psi_n\}$$

and we infer $\dim\ker F_x(0, \lambda_0) = \operatorname{codim} \operatorname{ran} F_x(0, \lambda_0) = 1$, so (S) holds. Moreover,

$$F_{x\lambda}(0, \lambda)[\psi_1] = -\psi_1 \notin \operatorname{ran} F_x(0, \lambda_0),$$

and (T) is satisfied. Theorem 4.3 ensures the existence of a unique continuous branch of nontrivial solutions of problem (2) bifurcating from $(0, \lambda_0)$. □

Problem 18:
Let $g \in C^2(\mathbb{R}\times\mathbb{R}\times\mathbb{R};\mathbb{R}), (x,z,\lambda) \mapsto g(x,z,\lambda)$ be 2π-periodic in x with

$$g(x,0,\lambda)=0, \quad g_z(x,0,\lambda)=0, \quad g_{z\lambda}(x,0,\lambda)=0 \quad \text{for all } x\in\mathbb{R}, \lambda\in\mathbb{R}.$$

In order to find nontrivial 2π-periodic solutions $u \in C^2(\mathbb{R})$ of the ODE

$$-u'' = \lambda u + g(\,\cdot\,,u,\lambda) \qquad \text{on } \mathbb{R} \tag{3}$$

in a neighborhood of $(u_0,\lambda_0)=(0,0)$, proceed as follows:

(a) Let $F: C^2_{\text{per}}(\mathbb{R})\times\mathbb{R} \to C_{\text{per}}(\mathbb{R}), F(u,\lambda) := u'' + \lambda u + g(\,\cdot\,,u,\lambda)$ where

$$C^k_{\text{per}}(\mathbb{R}) := \big\{u \in C^k(\mathbb{R})\colon u(x) = u(x+2\pi) \text{ for all } x\in\mathbb{R}\big\} \qquad \text{for } k\in\mathbb{N}_0.$$

Show that F is twice continuously Fréchet differentiable and calculate F' and F''.

(b) Show that $\ker(F_u(0,0)) = \operatorname{span}\{\mathbb{1}\}$; $\operatorname{ran}(F_u(0,0)) = \Big\{z\in C_{\text{per}}(\mathbb{R})\colon \int_0^{2\pi} z(t)\,\mathrm{d}t = 0\Big\}$.

(c) Prove that there exist $\delta>0$ and a continuous branch $(-\delta,\delta)\to C^2_{\text{per}}(\mathbb{R})\times\mathbb{R},\ s\mapsto(\hat u(s),\hat\lambda(s))$ with the property that

$$\Big\{(\hat u(s),\hat\lambda(s))\colon 0<|s|<\delta\Big\}$$

collects all nontrivial 2π-periodic solutions of problem (3) in a neighborhood of $(0,0)$.

Solution to problem 18:

(a) First let $k\in\mathbb{N}_0$. Recall that $C^k_b(x)$ consists of functions $f\in C^k(\mathbb{R})$ such that

$$\|f\|_{C^k_b(\mathbb{R})} = \sup\Big\{|f(x)|, |f'(x)|, \dots, \Big|f^{(k)}(x)\Big|\colon x\in\mathbb{R}\Big\} < \infty.$$

Since $\tau\colon C^k_{\mathrm{b}}(\mathbb{R})\to C^k_{\mathrm{b}}(\mathbb{R}), f\mapsto f(\,\cdot\,-2\pi)$ defines an isometric isomorphism, we see that $C^k_{\text{per}}(\mathbb{R}) = \ker(I-\tau)$ is a closed subspace of $C^k_b(\mathbb{R})$, and thus is a Banach space. We denote $C_{\text{per}}(\mathbb{R}) := C^0_{\text{per}}(\mathbb{R})$.

<u>*Claim:*</u> F is twice continuously differentiable and

$$\begin{aligned} F'(u,\lambda)[(h_u,h_\lambda)] &= h_u'' + h_\lambda u + \lambda h_u + g_z(\,\cdot\,,u,\lambda)h_u + g_\lambda(\,\cdot\,,u,\lambda)h_\lambda,\\ F''(u,\lambda)[(h_u,h_\lambda),(\eta_u,\eta_\lambda)] &= h_\lambda\eta_u + \eta_\lambda h_u + h_u\eta_u g_{zz}(\,\cdot\,,u,\lambda) + (\eta_\lambda h_u + h_\lambda\eta_u)g_{z\lambda}(\,\cdot\,,u,\lambda) + h_\lambda\eta_\lambda g_{\lambda\lambda}(\,\cdot\,,u,\lambda). \end{aligned}$$

<u>*Proof:*</u> We write $F = G+H$ with

$$\begin{aligned} G&\colon C^2_{\text{per}}(\mathbb{R})\times\mathbb{R}\to C_{\text{per}}(\mathbb{R}), G(u,\lambda) = g(\,\cdot\,,u,\lambda),\\ H&\colon C^2_{\text{per}}(\mathbb{R})\times\mathbb{R}\to C_{\text{per}}(\mathbb{R}), H(u,\lambda) = u''+\lambda u. \end{aligned}$$

Using the product rule we see that H is infinitely differentiable with

$$\begin{aligned} &H'(u,\lambda)[(h_u,h_\lambda)] = h_u'' + h_\lambda u + \lambda h_u,\\ &H''(u,\lambda)[(h_u,h_\lambda),(\eta_u,\eta_\lambda)] = h_\lambda\eta_u + \eta_\lambda h_u,\\ &H''' = 0, \dots \end{aligned}$$

In particular, H is twice continuously differentiable.

Next, we show that G is twice continuously differentiable and that

$$\begin{aligned} G'(u,\lambda)[(h_u,h_\lambda)] &= h_u g_z(\,\cdot\,,u,\lambda) + h_\lambda g_\lambda(\,\cdot\,,u,\lambda),\\ G''(u,\lambda)[(h_u,h_\lambda),(\eta_u,\eta_\lambda)] &= h_u\eta_u g_{zz}(\,\cdot\,,u,\lambda) + (\eta_\lambda h_u + h_\lambda\eta_u)g_{z\lambda}(\,\cdot\,,u,\lambda) + h_\lambda\eta_\lambda g_{\lambda\lambda}(\,\cdot\,,u,\lambda). \end{aligned}$$

To do this, we proceed similar to Problem 7. Let $(u,\lambda),(h_u,h_\lambda),(\eta_u,\eta_\lambda),(\rho_u,\rho_\lambda)\in C^2_{\text{per}}(\mathbb{R})\times\mathbb{R}$. We choose

$$K := [0,2\pi]\times[\inf u - 1, \sup u + 1]\times[\lambda-1,\lambda+1] \subseteq \mathbb{R}^3,$$

which is compact, so that $g, g_z, g_\lambda, \dots$ are uniformly continuous on K. Thus there exists an increasing function $\varphi\colon (0,\infty) \to [0,\infty]$ with $\lim_{r\to 0} \varphi(r) = 0$ so that $\left|g(x,u,\lambda) - g(\tilde{x},\tilde{u},\tilde{\lambda})\right| \le \varphi(|x-\tilde{x}| + |z-\tilde{z}| + \left|\lambda - \tilde{\lambda}\right|)$ for $(x,z,\lambda), (\tilde{x},\tilde{z},\tilde{\lambda}) \in K$; and the same estimate holds for derivatives of g.

We then calculate for $\|h_u\|_\infty, |h_\lambda| \le 1$:

$$\begin{aligned}
&\|G(u+h_u,\lambda+h_\lambda) - G(u,\lambda) - h_u g_z(\,\cdot\,,u,\lambda) - h_\lambda g_\lambda(\,\cdot\,,u,\lambda)\|_\infty \\
&= \left\| \int_0^1 h_u[g_z(\,\cdot\,,u+th_u,\lambda+th_\lambda) - g_z(\,\cdot\,,u,\lambda)] + h_\lambda[g_\lambda(\,\cdot\,,u+th_u,\lambda+th_\lambda) - g_\lambda(\,\cdot\,,u,\lambda)]\,\mathrm{d}t \right\|_\infty \\
&\le \int_0^1 \|h_u\|_\infty \varphi(|t|\|h_u\| + |t||h_\lambda|) + |h_\lambda|\varphi(|t|\|h_u\| + |t||h_\lambda|)\,\mathrm{d}t \\
&\le (\|h_u\|_\infty + |h_\lambda|)\varphi(\|h_u\|_\infty + |h_\lambda|) = \mathrm{o}(\|h_u\|_\infty + |h_\lambda|)
\end{aligned}$$

as $(h_u,h_\lambda) \to 0$. So G is Fréchet-differentiable and the derivative is as stated.

Next we confirm the formula for the second derivative. We calculate for $\|\eta_u\|_\infty, |\eta_\lambda| \le 1$:

$$\begin{aligned}
I &:= G'(u+\eta_u,\lambda+\eta_\lambda)[(h_u,h_\lambda)] - G'(u,\lambda)[(h_u,h_\lambda)] \\
&\quad - h_u\eta_u g_{zz}(\,\cdot\,,u,\lambda) - (\eta_\lambda h_u + h_\lambda\eta_u) g_{z\lambda}(\,\cdot\,,u,\lambda) - h_\lambda\eta_\lambda g_{\lambda\lambda}(\,\cdot\,,u,\lambda) \\
&= \int_0^1 h_u(\eta_u[g_{zz}(\,\cdot\,,u+t\eta_u,\lambda+t\eta_\lambda) - g_{zz}(\,\cdot\,,u,\lambda)] + \eta_\lambda[g_{z\lambda}(\,\cdot\,,u+t\eta_u,\lambda+t\eta_\lambda) - g_{z\lambda}(\,\cdot\,,u,\lambda)])\,\mathrm{d}t \\
&\quad + \int_0^1 h_\lambda(\eta_u[g_{\lambda z}(\,\cdot\,,u+t\eta_u,\lambda+t\eta_\lambda) - g_{\lambda z}(\,\cdot\,,u,\lambda)] + \eta_\lambda[g_{\lambda\lambda}(\,\cdot\,,u+t\eta_u,\lambda+t\eta_\lambda) - g_{\lambda\lambda}(\,\cdot\,,u,\lambda)])\,\mathrm{d}t
\end{aligned}$$

and thus get

$$\|I\|_\infty \le (\|h_u\|_\infty + |h_\lambda|)(\|\eta_u\|_\infty + |\eta_\lambda|)\varphi(\|\eta_u\|_\infty + |\eta_\lambda|) = (\|h_u\|_\infty + |h_\lambda|)\cdot \mathrm{o}(\|\eta_u\|_\infty + |\eta_\lambda|)$$

as $(\eta_u,\eta_\lambda) \to 0$. So G is twice Fréchet-differentiable and the derivative is as stated. For continuity of the second derivative, we calculate for $\|\rho_u\|_\infty, |\rho_\lambda| \le 1$:

$$\begin{aligned}
&\|G''(u+\rho_u,\lambda+\rho_\lambda)[(h_u,h_\lambda),(\eta_u,\eta_\lambda)] - G''(u,\lambda)[(h_u,h_\lambda),(\eta_u,\eta_\lambda)]\| \\
&\quad \le (\|h_u\|_\infty + |h_\lambda|)(\|\eta_u\|_\infty + |\eta_\lambda|)\varphi(\|\rho_u\|_\infty + |\rho_\lambda|) \\
&\quad = (\|h_u\|_\infty + |h_\lambda|)(\|\eta_u\|_\infty + |\eta_\lambda|)\cdot \mathrm{o}(1)
\end{aligned}$$

as $(\rho_u,\rho_\lambda) \to 0$. This shows that G is twice continuously Fréchet-differentiable.

The claim now follows from $F = G + H$ and linearity of the derivative operator. ☐

(b) <u>*Claim:*</u> $\ker(F_u(0,0)) = \mathrm{span}\{\mathbb{1}\}$.

<u>*Proof:*</u> From part (a) we know that

$$F_u(0,0)[h] = F'(0,0)[(h,0)] = h''$$

for $h \in C^2_{\mathrm{per}}(\mathbb{R})$. For such h, the following are equivalent:

$$\begin{aligned}
h \in \ker F_u(0,0) \quad &\iff \quad h'' = 0 \quad \iff \quad \exists \alpha,\beta \in \mathbb{R}\colon h(x) = \alpha x + \beta \text{ for } x \in \mathbb{R} \\
&\iff \quad \exists \beta \in \mathbb{R} : h(x) = \beta \text{ for } x \in \mathbb{R} \quad \iff \quad h \in \mathrm{span}\{\mathbb{1}\}
\end{aligned}$$

where, in passing from the first to the second line, we exploited that h is periodic. ☐

<u>*Claim:*</u> $\mathrm{ran}(F_u(0,0)) = \left\{ z \in C_{\mathrm{per}}(\mathbb{R})\colon \int_0^{2\pi} z(x)\,\mathrm{d}x = 0 \right\}$.

<u>*Proof:*</u> Assume $z \in \mathrm{ran}\,(F_u(0,0))$. Then, there exists $w \in C^2_{\mathrm{per}}(\mathbb{R})$ with $z = F_u(0,0)[w] = w''$ on $\mathbb{R}$. This yields, however,

$$\int_0^{2\pi} z(x)\,\mathrm{d}x = \int_0^{2\pi} w''(x)\,\mathrm{d}x = w'(2\pi) - w'(0) = 0$$

since w is periodic.

Conversely, assume that $z \in C_{\text{per}}(\mathbb{R})$ with $\int_0^{2\pi} z(x)\,\mathrm{d}x = 0$. We try to find $w \in C^2_{\text{per}}(\mathbb{R})$ with $z = F_u(0,0)[w] = w''$ on $\mathbb{R}$. Integration suggests the ansatz

$$\begin{aligned} w(x) &= \alpha + \beta x + \int_0^x \int_0^y z(t)\,\mathrm{d}t\mathrm{d}y, \qquad \text{hence} \\ w'(x) &= \beta + \int_0^x z(t)\,\mathrm{d}t \end{aligned}$$

with constants $\alpha = w(0)$ and $\beta = w'(0)$. By assumption on z, we have for $x \in \mathbb{R}$

$$w'(x+2\pi) - w'(x) = \int_0^{2\pi} z(x)\,\mathrm{d}x = 0,$$

i.e. w' has the asserted periodicity. Moreover, choosing $\beta := -\frac{1}{2\pi}\int_0^{2\pi}\int_0^y z(t)\,\mathrm{d}t\mathrm{d}y$, we also have, using that w' is 2π-periodic,

$$\begin{aligned} w(x+2\pi) - w(x) &= \int_x^{x+2\pi} w'(t)\,\mathrm{d}t = \int_0^{2\pi} w'(t)\,\mathrm{d}t \\ &= w(2\pi) - w(0) = 2\pi\beta + \int_0^{2\pi}\int_0^y z(t)\,\mathrm{d}t\mathrm{d}y \qquad = 0 \end{aligned}$$

and hence $w \in C^2_{\text{per}}(\mathbb{R})$ with $z = F_u(0,0)[w]$. □

(c) *Proof:* Finally, we intend to apply the Crandall-Rabinowitz Bifurcation Theorem.

From part (b) we defer that $F_u(0,0)\colon C^2_{\text{per}}(\mathbb{R}) \to C_{\text{per}}(\mathbb{R})$ is a (1,1)-Fredholm operator, meaning the simplicity assumption (S) from the lecture is satisfied. It remains to check the transversality condition

(T) $$F_{u\lambda}(0,0)[\phi] \notin \operatorname{ran}(F_x(0,0))$$

where $\phi = \mathbb{1}$. We calculate

$$F_{u\lambda}(0,0)[\phi] = F''(0,0)[(\mathbb{1},0),(0,1)] = \mathbb{1},$$

and using (b) find $F_{u\lambda}(0,0)[\phi] \notin \operatorname{ran}(F_u(0,0))$.

An application of Theorem 4.3 then closes the proof. □