# Problem Sheet 11
## Bifurcation Theory
Winter Semester 2022/23

23.1.2023

For $x, y \in \mathbb{C}^n$ we write $x \cdot y = \sum_{k=1}^n x_k y_k$ and $\langle x, y \rangle = \overline{x} \cdot y = \sum_{k=1}^n \overline{x_k} y_k$.

We recall assumptions (H1)–(H4):

(H1) "Regularity": $f \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ with $f(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$.

(H2) "Simplicity": There are $\phi, \psi \in \mathbb{C}^n$ with $\langle \psi, \phi \rangle = 1$ and $\beta > 0$ such that

$$\ker(f_x(\lambda_0, 0) - i\beta) = \operatorname{span}\{\phi\}, \quad \ker(f_x(\lambda_0, 0)^\top + i\beta) = \operatorname{span}\{\psi\}.$$

(H3) "Nonresonance": For all $k \in \mathbb{Z}$ such that $|k| \neq 1$ we have $\ker(f_x(\lambda_0, 0) - ik\beta) = \{0\}$.

(H4) "Transversality": $\operatorname{Re}(\langle f_{x\lambda}(\lambda_0, 0)\phi, \psi \rangle) \neq 0$.

## Problem 27 (Hopf bifurcation is incompatible with energy method):
Consider the pendulum equation

$$\text{(1)} \qquad \theta''(t) + \lambda \sin(\theta(t)) = 0$$

as discussed in Chapter 2. Write (1) as a first order problem

$$\text{(2)} \qquad x'(t) = f(\lambda, x(t))$$

for $x = (\theta, \theta')$ and show that assumptions (H1)–(H4) are not satisfied for any $\lambda_0 \in \mathbb{R}$.

## Problem 28 (On simplicity(H2)):
Let $\phi, \psi$ be as in (H2). Show that $\psi \cdot \phi = 0$.

## Problem 29:
Prove Proposition 7.1:

Let $\xi \in \mathbb{C}^n$, $k \in \mathbb{Z}$, $t \in \mathbb{R}$. Further let $\phi, \psi$ be as in (H2) and write $A := f_x(\lambda_0, 0)$. Then

$$y_k(t) := -\tfrac{1}{\beta} \int_0^t \exp\left(\frac{t - \tau}{\beta} A\right) \xi \, e^{ik\tau} \, d\tau$$

has the following closed form representation:

(i) $y_k(t) = \left(e^{ikt} I - e^{\frac{t}{\beta} A}\right)(A - ik\beta)^{-1}\xi$ if $|k| \neq 1$,

(ii) $y_1(t) = \left(e^{it} I - e^{\frac{t}{\beta} A}\right)w - \tfrac{t}{\beta} e^{it}\langle \psi, \xi \rangle \phi$

   where $w \in \mathbb{C}^n$ satisfies $(A - i\beta)w = \xi - \langle \psi, \xi \rangle \phi$.