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Boundary and Eigenvalue Problems

6th Exercise Sheet

16) Prove: a function $u \in C^2(\bar{\Omega}), u|_{\Gamma_0} = 0$, solving the boundary value problem

$$-\mathrm{div}(A\nabla u) + b\nabla u + cu = r \,\mathrm{in}\, \Omega,$$
$$u = 0 \text{ on } \Gamma_0,\ \frac{\partial u}{\partial \mu} + \gamma u = 0 \text{ on } \Gamma_1$$

in the weak sense, is in particular a classical solution (meaning that u satisfies the differential equation and the boundary conditions in the pointwise sense).

17) Let $\Omega = \{(x, y, z) \in \mathbb{R}^3 : x > 0, 0 < y^2 + z^2 < e^{-\frac{1}{x}}\}$. Show that

$$v(x, y, z) = \begin{cases} 1 & : \quad y = z = 0 \\ -x \log(y^2 + z^2) & : \quad \text{else} \end{cases}$$

is in $C^2(\Omega)$ and solves the boundary value problem

$$\begin{cases} -\Delta u &= 0 \quad \mathrm{in}\, \Omega \\ \quad u &= 1 \quad \mathrm{on}\, \partial\Omega \end{cases},$$

but $u \notin C(\bar{\Omega})$. Is Ω a Lipschitz domain?

18) Let B be a bilinear form satisfying the requirements of the Lax-Milgram lemma. In the lecture, the existence of a bounded linear operator $A : H \to H$ such that

$$\langle Au, v\rangle = B[u, v] \quad (u, v \in H)$$

was shown. Give an alternative proof of the lemma by first showing that $A(H)$ is closed and subsequently that $A(H)^{\perp} = \{0\}$.
Hint: We have $c_0||u|| \leq ||Au||$.