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Boundary and Eigenvalue Problems
3rd Exercise Sheet

7) a) Let $L[u] = (-\Delta)^k u = \underbrace{(-\Delta) \circ \cdots \circ (-\Delta)}_{k-\text{times}} u$. Show that L is uniformly strongly elliptic.

b) Show that

$$L[u] = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$$

is a quasilinear elliptic differential operator.

8) Let $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$ and consider the boundary value problem

$$(1) \qquad \begin{cases} u_{xy} = 0 & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

for $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and given $u_0 \in C(\partial\Omega)$. Prove that the relation

$$u(0,0) + u(1,1) = u(1,0) + u(0,1)$$

holds for any solution of (1) and conclude that there exists a $u_0 \in C(\partial\Omega)$ such that (1) has no solution. Is the operator $L[u] = u_{xy}$ elliptic?

9) Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set with Lipschitz-boundary. Verify for $u, v \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ the (generalized) Green's identities

$$\text{a)} \qquad \int_\Omega L[u] v dx = -\int_{\partial\Omega} \frac{\partial u}{\partial \mu} v d\sigma + \int_\Omega (\nabla u)^T A \nabla v dx$$

$$\text{b)} \qquad \int_\Omega (L[u]v - uL[v]) dx = \int_{\partial\Omega} \left(\frac{\partial v}{\partial \mu} u - v \frac{\partial u}{\partial \mu}\right) dx$$

where $L[u] = -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial u}{\partial x_j}\right)$ $\quad (a_{ij} \in C^1(\bar{\Omega}), \quad A(x) = (a_{ij}(x))$ symmetric matrix).