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Boundary and Eigenvalue Problems

11th Exercise Sheet

32) Let $R,\ S$ be Banach spaces and denote by $\mathcal{B}(R,S)$ and $\mathcal{K}(R,S)$ the spaces of all bounded linear resp. all compact linear operators from R to S, each of them endowed with the usual operator norm $||A|| := \sup_{x\neq 0} \frac{||Ax||}{||x||}$ ($A \in \mathcal{B}(R,S)$, resp. $A \in \mathcal{K}(R,S)$). Show:

(a) $\mathcal{B}(R,S)$ is a Banach space.

(b) $\mathcal{K}(R,S)$ is a closed subspace of $\mathcal{B}(R,S)$.

33) Let $k \in L_2((0,1)\times(0,1))$ and

$$(Ku)(x) := \int_0^1 k(x,y)u(y)\,dy \qquad (u \in L_2(0,1),\ x \in (0,1)).$$

Show that $K : L_2(0,1) \longrightarrow L_2(0,1)$ is compact.

Hint: Approximate k by step functions k_N (i.e. piecewise constant functions). Show that the operators

$$(K_N u)(x) := \int_0^1 k_N(x,y)u(y)\,dy \qquad (u \in L_2(0,1),\ x \in (0,1),\ N \in \mathbb{N}).$$

are compact.

34) Prove that $L^\infty(a,b)$ is not separable.

35) Let $(\varphi_k)_{k\in\mathbb{N}}$ and $(\psi_j)_{j\in\mathbb{N}}$ be complete orthonormal systems in $L^2(0,1)$. Define ω_{kj} by

$$\omega_{kj}(x,y) = \varphi_k(x)\psi_j(y).$$

Show that $(\omega_{kj})_{k,j\in\mathbb{N}}$ is a complete orthonormal system in $L^2((0,1)\times(0,1))$.