Exercise 32. By Hardy's inequality we have

$$\int\int \frac{|\psi(x)|^2|\psi(y)|^2}{|x-y|}dydx = \int |\psi(x)|^2\Big(\int \frac{|\psi(y)|^2}{|x-y|}dy\Big)dx = \int |\psi(x)|^2\Big(\int \frac{|\psi(x+y)|^2}{|y|}dy\Big)dx \leq$$

$$\|\nabla\psi\|_2\|\psi\|_2 \int |\psi(x)|^2 dx = \|\nabla\psi\|_2\|\psi\|_2^3$$

which finishes the proof.

□

Exercise 33. We first observe that if $\psi \in H^1(\mathbb{R}^3)$ with $\|\psi\|_2 = 1$ then the function given by $\psi_\alpha(x) = \alpha^{\frac{3}{2}}\psi(\alpha x)$ is also in $H^1(\mathbb{R}^3)$ with $\|\psi_\alpha\|_2 = 1$. Hence,

$$P_1^\alpha(\psi_\alpha) = \int |\nabla\psi_\alpha(x)|^2dx - C\alpha\int\int \frac{|\psi_\alpha(x)|^2|\psi_\alpha(y)|^2}{|x-y|}dxdy =$$

$$\alpha^3\int |\nabla[\psi(\alpha x)]|^2dx - C\alpha^7\int\int \frac{|\psi(\alpha x)|^2|\psi(\alpha y)|^2}{|x-y|}dxdy =$$

$$\alpha^5\int |(\nabla\psi)(\alpha x)|^2dx - C\alpha^7\int\int \frac{|\psi(\alpha x)|^2|\psi(\alpha y)|^2}{|x-y|}dxdy =$$

$$\alpha^2\int |\nabla\psi(x)|^2dx - C\alpha^2\int\int \frac{|\psi(x)|^2|\psi(y)|^2}{|x-y|}dxdy = \alpha^2 P_1^1(\psi)$$

where from the third line to the fourth line we made the change of variables $\alpha x \to x$ and $\alpha y \to y$. Therefore, we have shown that $\mathcal{E}_1^\alpha = \alpha^2\mathcal{E}_1^1$ which is the desired equality.

□

Exercise 34. The proof is the same as in the previous exercise but we work with the functional (instead of P_1^α)

$$P_n^{v\alpha,\alpha}(\psi) = \sum_{j=1}^n \|\nabla_j\psi\|_2^2 + \Big\langle\psi, \sum_{1\leq i<j\leq n}\frac{v\alpha}{|x_i-x_j|}\psi\Big\rangle - C\alpha\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{\rho(x)\rho(y)}{|x-y|}\,dxdy$$

where

$$\rho(x) = \sum_{j=1}^n \int_{\mathbb{R}^{3(n-1)}} \Big|\psi(x_1,\ldots,x_{j-1},x,x_{j+1}\ldots,x_n)\Big|^2\, dx_1\ldots dx_{j-1}\hat{dx_j}dx_j\ldots dx_n.$$

The function ψ_α that we use in this case is given by the formula

$$\psi_\alpha(x_1,\ldots,x_n) = \alpha^{\frac{3}{2}n}\psi(\alpha x_1,\ldots,\alpha x_n).$$

□

Exercise 35. Notice that the following calculations work for general $f \in L^2(\mathbb{R}^3)$.

We know that for all $g \in L^2(\mathbb{R}^3)$ and $\phi \in D(N^{\frac{1}{2}})$ (where N is the number operator in the Fock space $\mathcal{F}(L^2(\mathbb{R}^3))$) we have

$$\|a(g)\phi\| \leq \|g\|_2\|N^{\frac{1}{2}}\phi\|. \tag{1}$$

It is straightforward to see that $\eta \in D(N^{\frac{1}{2}})$ since the series

$$\sum_{n=0}^{\infty} \frac{\sqrt{n}\ \|f\|_2^n}{\sqrt{n!}} < \infty.$$

In addition, we know that the operator $N^{\frac{1}{2}}$ is self-adjoint and hence closed. Thus, from (1) and the fact that the sequence

$$\Big\{ N^{\frac{1}{2}} \Big(\overbrace{\sum_{n=0}^{k} \frac{f^{\otimes n}}{\sqrt{n!}}}^{=\eta_k} \Big) \Big\}_k$$

is Cauchy in the norm $\|\cdot\|$ we derive that the series

$$\Big\{ a(g) \Big(\sum_{n=0}^{k} \frac{f^{\otimes n}}{\sqrt{n!}} \Big) \Big\}_k = \Big\{ \sum_{n=0}^{k} \frac{a(g) f^{\otimes n}}{\sqrt{n!}} \Big\}_k$$

is convergent in $\|\cdot\|$ for all $g \in L^2(\mathbb{R}^3)$. Now observe that

$$\|a(g)(\eta_k - \eta)\| \le \|g\|_2 \|N^{\frac{1}{2}}(\eta_k - \eta)\| \to 0 \text{ as } k \to \infty.$$

Hence, we can exchange $a(g)$ with the series and obtain

$$a(g)\eta = a(g) \sum_{n=0}^{\infty} \frac{f^{\otimes n}}{\sqrt{n!}} = \sum_{n=0}^{\infty} \frac{a(g) f^{\otimes n}}{\sqrt{n!}} = \sum_{n=0}^{\infty} \sqrt{n}\ \frac{\langle g, f\rangle f^{\otimes(n-1)}}{\sqrt{n!}} = \tag{2}$$

$$\langle g, f\rangle \sum_{n=1}^{\infty} \frac{f^{\otimes(n-1)}}{\sqrt{(n-1)!}} = \langle g, f\rangle \eta$$

which is equivalent to $a(k)\eta = f(k)\eta$ since $a(g) = \int a(k) g(k)$ by definition.

$\square$