Exercise 1, Question 1. Since $(L^2(\mathbb{R}^{3N}), \|\cdot\|_2)$ is a Hilbert space and $L^2_a(\mathbb{R}^{3N}) \subset L^2(\mathbb{R}^{3N})$, in order to show that $(L^2_a(\mathbb{R}^{3N}), \|\cdot\|_2)$ is also a Hilbert space, it suffices to prove that $L^2_a(\mathbb{R}^{3N})$ is closed in $(L^2(\mathbb{R}^{3N}), \|\cdot\|_2)$.

Hence, consider a sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^2_a(\mathbb{R}^{3N})$, an $f \in L^2(\mathbb{R}^{3N})$ and suppose that $\|f_n - f\|_2 \to 0$ as $n \to \infty$. Fix a permutation $\sigma \in S_N$. By the definitions we know that

$$T_\sigma f_n = (-1)^\sigma f_n. \tag{1}$$

Therefore, if the operator $T_\sigma : L^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ is bounded the LHS of (1) converges to $f(x_{\sigma(1)}, \dots, x_{\sigma(N)})$ and the RHS of (1) to $(-1)^\sigma f(x_1, \dots, x_N)$ which is exactly what we want. But it is trivial to see that $\|T_\sigma \psi\|_2 = \|\psi\|_2$, for all $\psi \in L^2(\mathbb{R}^{3N})$ and the proof is complete.

□

Exercise 1, Question 2. Fix $\tilde{\sigma} \in S_N$. By definition

(2)
$$T_{\tilde{\sigma}} P_{a,N} = T_{\tilde{\sigma}} \Big(\frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma T_\sigma \Big) = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma T_{\tilde{\sigma}} T_\sigma = (-1)^{\tilde{\sigma}} \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^{\tilde{\sigma}+\sigma} T_{\tilde{\sigma}\circ\sigma} =$$

$$(-1)^{\tilde{\sigma}} P_{a,N}.$$

Since this is true for all $\tilde{\sigma} \in S_N$ we obtain $P_{a,N}(L^2(\mathbb{R}^{3N})) \subset L^2_a(\mathbb{R}^{3N})$. Observe also that by the definition of the $L^2_a(\mathbb{R}^{3N})$ space we have that $P_{a,N}|_{L^2_a} = Id_{L^2_a}$.

By a change of variables it is straightforward to see that for all $\psi \in L^2(\mathbb{R}^{3N})$ we have $\langle \psi, T_\sigma \psi \rangle = \langle T_{\sigma^{-1}} \psi, \psi \rangle$, i.e. $(T_\sigma)^* = T_{\sigma^{-1}}$. Hence,

$$(P_{a,N})^* = \Big(\frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma T_\sigma \Big)^* = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma (T_\sigma)^* = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma T_{\sigma^{-1}} = \tag{3}$$

$$\frac{1}{N!} \sum_{\sigma \in S_N} (-1)^{\sigma^{-1}} T_{\sigma^{-1}} = P_{a,N}$$

which means that $P_{a,N}$ is self-adjoint.

Finally, with the use of (2) we observe that

$$P_{a,N} P_{a,N} = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma T_\sigma P_{a,N} = \frac{1}{N!} \sum_{\sigma \in S_N} (-1)^\sigma (-1)^\sigma P_{a,N} = P_{a,N}$$

and the proof is complete.

□

Exercise 1, Question 3. By the definition of the Hamiltonian operator $H_{N,Z}$ we have that $H_{N,Z}$ is invariant under all permutations $\sigma \in S_N$. Therefore, it is obvious that $H_{N,Z}\psi$ inherits the symmetry properties of the input function ψ. In other words, if $\psi \in L^2_a(\mathbb{R}^{3N})$ then $H_{N,Z}\psi \in L^2_a(\mathbb{R}^{3N})$ which completes the argument.

□

Exercise 1, Question 4. By Exercise 13 of last semester we know that the operator $H_{N,Z} : H^2(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ is self-adjoint. In the previous question we showed that the Hilbert space $(L^2_a(\mathbb{R}^{3N}), \|\cdot\|_2)$ is invariant under the action of the operator $H_{N,Z}$ and by the self-adjointness we also obtain that the space $(L^2_a(\mathbb{R}^{3N}))^{\perp}$ is invariant under $H_{N,Z}$. Thus,

$$Ran(H_{N,Z}|_{L^2_a} \pm i) = L^2_a(\mathbb{R}^{3N}).$$

By the basic criterion of self-adjointness the proof is complete.

□

Exercise 1, Question 5. The proof is very similar to the proof of the HVZ theorem given in last semester. For this reason we will try to highlight the changes that need to be made.

Let us denote by $E_{N-1} = \inf \sigma(H_{N-1}|_{L^2_a})$. The argument consists of two steps. The first one is the following:

We will prove that

$$[E_{N-1}, \infty) \subset \sigma(H_N|_{L^2_a}). \tag{4}$$

To this direction, let $\lambda = E_{N-1} + \delta$ for some $\delta \geq 0$ and write

$$H_N = H_{N-1} - \Delta_{x_N} - \frac{Z}{|x_N|} + \sum_{i=1}^{N-1} \frac{1}{|x_i - x_N|}. \tag{5}$$

Then, for a given $\epsilon > 0$ there exist $\psi_{N-1} \in C_c^{\infty}(\mathbb{R}^{3(N-1)}) \cap L^2_a(\mathbb{R}^{3(N-1)})$ and $\phi \in C_c^{\infty}(\mathbb{R}^3)$ with $\|\psi_{N-1}\|_2 = \|\phi\|_2 = 1$ and

$$\|(H_{N-1} - E_{N-1})\psi_{N-1}\|_2 < \frac{\epsilon}{3}, \quad \|(-\Delta_{x_N} - \delta)\phi\|_2 < \frac{\epsilon}{3}. \tag{6}$$

Let us denote by $\phi_h(\cdot) = \phi(\cdot - h)$, for $h \in \mathbb{R}^3$. Trivially, $\|\psi_{N-1} \otimes \phi_h\|_2 = 1$. We would like to have that $\psi_{N-1} \otimes \phi_h \in L^2_a(\mathbb{R}^{3N})$, but this is not true in general. So we have to consider the projection $P_{a,N}(\psi_{N-1} \otimes \phi_h)$ which since $\psi \in L^2_a(\mathbb{R}^{3(N-1)})$ we obtain the expression (7)

$$P_{a,N}(\psi_{N-1} \otimes \phi_h) = \frac{1}{N}\Big(\overbrace{\psi_{N-1}(x_1, \ldots, x_{N-1})\phi_h(x_N)}^{=f_1} - \overbrace{\psi_{N-1}(x_1, \ldots, x_{N-2}, x_N)\phi_h(x_{N-1})}^{=f_2}$$
$$- \overbrace{\phi_{N-1}(x_1 \ldots, x_{N-3}, x_N, x_{N-1})\phi_h(x_{N-2})}^{=f_{N-1}} - \ldots - \overbrace{\psi_{N-1}(x_N, x_2, \ldots, x_{N-1})\phi_h(x_1)}^{f_N}\Big).$$

Notice that since all the summands of the RHS of (7) have disjoint supports we have

$$\|\sqrt{N} P_{a,N}(\psi_{N-1} \otimes \phi_h)\|_2^2 = \frac{1}{N}\sum_{i=1}^{N} \|f_i\|_2^2 = \frac{1}{N}\sum_{i=1}^{N} 1 = 1. \tag{8}$$

Similarly, since the operators $P_{a,N}$ and H_N commute we may write

$$\|(H_N - \lambda)\sqrt{N} P_{a,N}(\psi_{N-1} \otimes \phi_h)\|_2 = \|\sqrt{N} P_{a,N}(H_N - \lambda)\psi_{N-1} \otimes \phi_h\|_2 = \tag{9}$$
$$\|(H_N - \lambda)\psi_{N-1} \otimes \phi_h\|_2.$$

But then we proceed identically as in the proof of the classical HVZ theorem, i.e. by (5) and the triangle inequality we have

$$
\begin{aligned}
(10)\quad & \|(H_N-\lambda)\psi_{N-1}\otimes\phi_h\|_2 \le \|(H_{N-1}-E_{N-1})\psi_{N-1}\otimes\phi_h\|_2+\|(-\Delta_{x_N}-\delta)\psi_{N-1}\otimes\phi_h\|_2+ \\
& \Big\|\Big(-\frac{Z}{|x_N|}+\sum_{i=1}^{N-1}\frac{1}{|x_i-x_N|}\Big)\psi_{N-1}\otimes\phi_h\Big\|_2 = \|(H_{N-1}-E_{N-1})\psi_{N-1}\|_2+\|(-\Delta_{x_N}-\delta)\phi_h\|_2+ \\
& \Big\|\Big(-\frac{Z}{|x_N|}+\sum_{i=1}^{N-1}\frac{1}{|x_i-x_N|}\Big)\psi_{N-1}\otimes\phi_h\Big\|_2 < \frac{2\epsilon}{3}+\Big\|\Big(-\frac{Z}{|x_N|}+\sum_{i=1}^{N-1}\frac{1}{|x_i-x_N|}\Big)\psi_{N-1}\otimes\phi_h\Big\|_2 \\
& \le \frac{2\epsilon}{3}+\frac{Z}{dist(supp\phi_h,0)}+\frac{N-1}{dist(supp\phi_h,A_i)}<\epsilon
\end{aligned}
$$

for large enough h, where $A_i=\{x_i\in\mathbb{R}^3:\psi_{N-1}(\cdot,\dots,x_i,\dots,\cdot)\neq 0\}$. Hence, $\lambda\in\sigma(H_N|_{L^2_a})$ as desired.

The second and final step of the proof consists of showing that $\inf\sigma_{ess}(H_N|_{L^2_a})\ge E_{N-1}$. As in the classical proof of the HVZ theorem given last semester we use the IMS localization formula

$$
H_N=\sum_{k=0}^{N}J_{k,R}H_NJ_{k,R}-\sum_{k=0}^{N}|\nabla J_{k,R}|^2 \tag{11}
$$

which implies

$$
P_{a,N}H_NP_{a,N}=\sum_{k=0}^{N}P_{a,N}J_{k,R}H_NJ_{k,R}P_{a,N}-\sum_{k=0}^{N}P_{a,N}|\nabla J_{k,R}|^2P_{a,N}. \tag{12}
$$

The term $P_{a,N}J_{0,R}H_NJ_{0,R}P_{a,N}$ is treated exactly as in the classical proof after observing that $P_{a,N}$ and $J_{0,R}$ commute with each other and thus, the operator H_N acts on $L^2_a(\mathbb{R}^{3N})$. For the term $P_{a,N}J_{N,R}H_NJ_{N,R}P_{a,N}$ we do the following (similar considerations apply to the remaining terms $P_{a,N}J_{k,R}H_NJ_{k,R}P_{a,N}$ for $k=1,\dots,N-1$)

$$
\begin{aligned}
(13)\quad & P_{a,N}J_{N,R}H_NJ_{N,R}P_{a,N}= \\
& P_{a,N}P_{a,N-1}J_{N,R}\Big(H_{N-1}-\Delta_{x_N}-\frac{Z}{|x_N|}+\sum_{i=1}^{N-1}\frac{1}{|x_i-x_N|}\Big)J_{N,R}P_{a,N-1}P_{a,N}
\end{aligned}
$$

where we used the fact that $P_{a,N}P_{a,N-1}=P_{a,N}$. Next observe that the operators $P_{a,N-1}$ and $J_{N,R}$ commute since they act on different variables which allows us to rewrite the last expression as

$$
P_{a,N}J_{N,R}P_{a,N-1}\Big(H_{N-1}-\Delta_{x_N}-\frac{Z}{|x_N|}+\sum_{i=1}^{N-1}\frac{1}{|x_i-x_N|}\Big)P_{a,N-1}J_{N,R}P_{a,N}.
$$

Notice that the operator H_{N-1} acts now on the space $L^2_a(\mathbb{R}^{3(N-1)})$. Then proceed identically as in the proof of the classical HVZ theorem presented last semester.

The argument with the Weyl sequence goes through as it is and the last thing we have to notice is that the operator $J_{0,R}$ leaves the space $L^2_a(\mathbb{R}^{3N})$ invariant and so $J_{0,R}$ being compact from $H^1(\mathbb{R}^{3N}) \to L^2(\mathbb{R}^{3N})$ implies that it is compact as an operator from $H^1(\mathbb{R}^{3N}) \cap L^2_a(\mathbb{R}^{3N}) \to L^2_a(\mathbb{R}^{3N})$.

$\square$