(12)

$$\mathcal{F}\phi = (2\pi)^{-d/2} \int e^{-i\xi\cdot x}\phi(x)\,dx$$

(a)

$$f(p)\phi = \mathcal{F}^{-1}\Big[\underbrace{f(\xi)}_{\in L^\infty}\ \underbrace{\mathcal{F}\phi}_{\in L^2}\Big] \qquad (\phi\in L^2(\mathbb{R}^d)).$$

Recall that, $\mathcal{F}^{-1}\omega = L^2\text{-}\lim_{R\to\infty} (2\pi)^{-d/2}\int_{B_R} e^{+i\cdot\xi}\,\omega(\xi)\,d\xi$

$$\mathcal{F}\phi = L^2\text{-}\lim_{R\to\infty} (2\pi)^{-d/2}\int_{B_R} e^{+i\cdot x}\phi(x)\,dx$$

It follows that $f(\cdot)\,\mathcal{F}\phi = L^2\text{-}\lim_{R\to\infty} (2\pi)^{-d/2}\int_{B_R} e^{-i\cdot x}\phi(x)f(\xi)\,dx$

Now

$$(2\pi)^{-d/2}\int_{B_{R_1}} e^{ix\cdot\xi}\,(2\pi)^{-d/2}\left(\int_{B_{R_2}} e^{-i\xi\cdot y} f(\xi)\,\phi(y)\,dy\right)d\xi$$

$$= (2\pi)^{-d}\int_{B_{R_1}}\left(\int_{B_{R_2}} e^{+i\xi\cdot(x-y)} f(\xi)\,d\xi\right)\phi(y)\,dy \qquad (*)$$

Now:

(1) $$\int_{B_{R_2}} e^{+i\xi\cdot(x-y)} f(\xi)\,d\xi \xrightarrow[L^2(\mathbb{R}^d)]{} (2\pi)^{d/2}\mathcal{F}^{-1}(x-y)$$

Hence by Cauchy-Schwarz:

$$(*) \underset{R_1\to\infty}{\Longrightarrow} (2\pi)^{-d/2}\int_{B_{R_1}} e^{ix\xi}(2\pi)^{-d/2}\,\mathcal{F}\phi\, f(\xi)\,d\xi$$

$$= (2\pi)^{-d/2}\int_{B_{R_1}} \mathcal{F}^{-1}f(x-y)\,\phi(y)\,dy \qquad (**)$$

Now:

- $(\mathcal{F}^{-1}f)(x-\cdot)\,\phi(\cdot) \in L^1(\mathbb{R}^d)$ for each $x\in\mathbb{R}^d$ by Cauchy-Schwarz (since $(\mathcal{F}^{-1}f)(x-\cdot)\in L^2(\mathbb{R}^d)$, $\phi\in L^2(\mathbb{R}^d)$). Hence $\int_{B_{R_1}}(\mathcal{F}^{-1}f)(x-y)\phi(y)\,dy \xrightarrow[R_1\to\infty]{} \int_{\mathbb{R}^d}(\mathcal{F}^{-1}f)(x-y)\phi(y)\,dy$

- $(\mathcal{F}\phi)f\in L^2(\mathbb{R}^d) \Rightarrow \underset{R_1\to\infty}{L^2\text{-}\lim}\int_{B_{R_1}} e^{ix\xi}\,\mathcal{F}\phi\, f(\xi)\,d\xi$ exists

and $=$ const. $\mathcal{F}^{-1}[f\,\mathcal{F}\phi]$.

Hence:

$$\mathcal{F}^{-1}[f\,\mathcal{F}\phi] = f(p)\,\phi = \underbrace{(2\pi)^{-d/2}\int_{\mathbb{R}^d}(\mathcal{F}^{-1}f)(x-y)\,\phi(y)\,dy}_{\text{ordinary Lebesgue-integral for all } x\in\mathbb{R}^d.}$$

(b)

$$\left(g(x) f(p) \phi\right)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \underbrace{g(x)\left(\mathcal{F}^{-1} f\right)(x-y)}_{=: k(x,y)} \phi(y)\, dy$$

<u>claim:</u> $\displaystyle\int_{\mathbb{R}^d \times \mathbb{R}^d} |k(x,y)|^2 \, d(x,y) < \infty$

Thus, $g(x) f(p)$ is a Hilbert-Schmidt operator.

<u>proof:</u>

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} |k(x,y)|^2 \, d(x,y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |g(x)|^2 \left|\mathcal{F}^{-1} f(x-y)\right|^2 dy\, dx$$

$$= \int_{\mathbb{R}^d} |g(x)|^2 \underbrace{\left(\int_{\mathbb{R}^d} \left|\mathcal{F}^{-1} f(x-y)\right|^2 dy \right)}_{= \|f\|^2_{L^2(\mathbb{R}^d)}} dx$$

$$= \underbrace{\|f\|^2_{L^2}}_{< \infty, \text{ since } f \in L^2 \cap L^\infty.} \|g\|^2_{L^2} < \infty .$$

(13)

Corollary from Stone's theorem: Let $\{U(t)\}_{t\in\mathbb{R}}$ be a unitary group and $A: D(A) \to \mathcal{H}$ the generator. Moreover, let $E \subseteq D(A)$ (E dense in $\mathcal{H}$), $U(t)E \subseteq E$ ($\forall t \in \mathbb{R}$), then $A|_E$ is essentially self-adjoint on E.

(a) $(R_\varphi u)(x_1, x_2, x_3) = u(x_1 \cos\varphi - x_2 \sin\varphi, x_2\cos\varphi + x_1 \sin\varphi, x_3)$

- easy to show: $\{R_\varphi\}_{\varphi\in\mathbb{R}}$ is a unitary group.
(note: $\|R_\varphi u\|_{L^2(\mathbb{R}^3)} = \|u\|_{L^2(\mathbb{R}^3)}$, $R_{\varphi+\psi} = R_\varphi R_\psi$)

To prove strong continuity, it is sufficient to show $\|R_\varphi u - u\|_{L^2} \to 0$ as $\varphi \to 0$, for fixed $u \in L^2(\mathbb{R}^3)$.

Let $\varepsilon > 0$ be given. Chose $u_\varepsilon \in \underbrace{C_0^\infty(\mathbb{R}^3)}_{\text{dense in } L^2(\mathbb{R}^3)}$ s.t. $\|u_\varepsilon - u\| \leq \varepsilon/2$.

Then
$$\|R_\varphi u - u\|_{L^2} \leq \|R_\varphi u - R_\varphi u_\varepsilon\| + \|R_\varphi u_\varepsilon - u_\varepsilon\| + \|u_\varepsilon - u\| \underset{R_\varphi \text{ unitary}}{=} 2\|u - u_\varepsilon\| + \|R_\varphi u_\varepsilon - u_\varepsilon\|$$

Since $u_\varepsilon \in C_0^\infty$, it is easy to show that $\|R_\varphi u_\varepsilon - u_\varepsilon\| \to 0$ as $\varphi \to 0$. Hence there exists a $\varphi_0 > 0$ s.t. $\forall |\varphi| < \varphi_0$: $\|R_\varphi u_\varepsilon - u_\varepsilon\| \leq \frac{\varepsilon}{2}$.

$\Rightarrow \forall_{\varepsilon>0} \exists \varphi_0(\varepsilon) \forall_{|\varphi|<\varphi_0}$: $\|R_\varphi u - u\| \leq \varepsilon$.

(b)

Let $\psi \in D$, i.e. $\psi = e^{-|x|^2/2} \underbrace{P(x_1, x_2, x_3)}_{\text{Polynomial in } (x_1, x_2, x_3)}$.

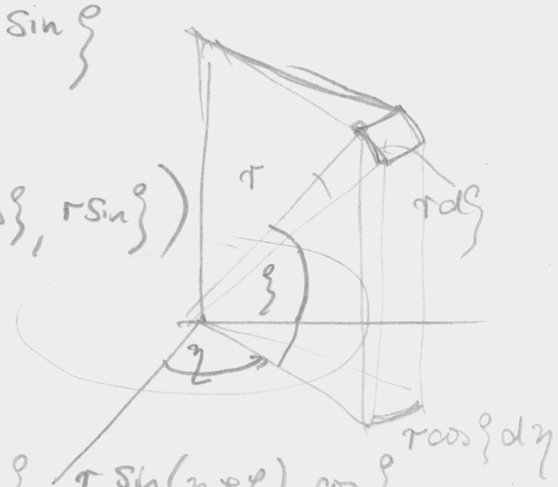
We show that $\{R_\varphi\}$ is strongly differentiable on D.

<u>claim</u>: $\left\| \frac{R_\varphi \psi - \psi}{\varphi} - i L_3 \psi \right\|_{L^2} \to 0 \quad (\varphi \to 0)$

proof:

For $u \in L^2(\mathbb{R}^3)$, let $\tilde{u}(r, \eta, \xi)$ be the representation of u in polar coordinates:

$$x_1 = r \cos\eta \cos\xi$$
$$x_2 = r \sin\eta \cos\xi$$
$$x_3 = r \sin\xi$$



$$\tilde{\psi}(r, \eta, \xi) = e^{-r^2/2} P(r\cos\eta\cos\xi, r\sin\eta\cos\xi, r\sin\xi)$$

$$\Rightarrow \widetilde{(R_\varphi \psi)}(r, \eta, \xi) = e^{-r^2/2} P\big(r\cos(\eta+\varphi)\cos\xi, r\sin(\eta+\varphi)\cos\xi, r\sin\xi\big)$$

$$\frac{\partial}{\partial\varphi} \widetilde{R\psi} = -e^{-r^2/2} \frac{\partial P}{\partial x_1}(\dots)\, r \sin(\eta+\varphi)\cos\xi$$
$$+ e^{-r^2/2} \frac{\partial P}{\partial x_2}(\dots)\, r\cos(\eta+\varphi)\cos\xi$$

A calculation shows:

$$\Rightarrow \frac{\partial}{\partial\varphi}\widetilde{R_\varphi\psi}\Big|_{\varphi=0} = \widetilde{x_1\left(\frac{\partial}{\partial x_2}\psi\right) - x_2\left(\frac{\partial}{\partial x_1}\psi\right)}$$

$$= i\,\widetilde{L_3\psi}$$

$$\left\|\frac{R_\varphi\psi-\psi}{\varphi} - iL_3\psi\right\|_{L^2}^2$$

$$= \int_{\mathbb{R}^3}\ldots\, d(x_1,x_2,x_3) = \int_0^\infty\int_{-\pi}^{\pi}\int_0^{2\pi}\left|\widetilde{\frac{R_\varphi\psi-\psi}{\varphi}} - i\widetilde{L_3\psi}\right|^2 r^2\cos\xi\, dr\, d\xi\, d\varphi$$

$$= \int_0^\infty\int_{-\pi}^{\pi}\int_0^{2\pi}\Bigg|\underbrace{\frac{\widetilde{R_\varphi\psi}-\tilde\psi}{\varphi} - \frac{\partial}{\partial\varphi}\widetilde{R_\varphi\psi}\Big|_{\varphi=0}}_{\to 0 \text{ for } (r,\eta,\xi) \text{ fixed } (\psi \text{ smooth})}\Bigg|^2 r^2\cos\xi\, dr\, d\xi\, d\varphi$$

$$\left|\frac{\partial}{\partial\varphi}\widetilde{R_\varphi\psi}\right| \le e^{-r^2/2}\times\left(\text{polynomial in } r\right)$$

So $\left|\dfrac{\widetilde{R_\varphi}-\tilde\psi}{\varphi}\right| \le e^{-r^2/2}\times(\text{polynomial in } r)$

Use Lebesgue dominated convergence theorem to show

that $\left\|\dfrac{R_\varphi\psi-\psi}{\varphi} - iL_3\psi\right\| \to 0 \quad (\varphi\to 0)$

(c)

We show that $R_\varphi D \subseteq D$.

This follows from the following. Suppose $P(x_1, x_2, x_3) = x_1^{k_1} x_2^{k_2} x_3^{k_3}$, then

$$P(x_1 \cos\varphi - x_2 \sin\varphi,\ x_2 \cos\varphi + x_1 \sin\varphi,\ x_3)$$

$$= (x_1 \cos\varphi - x_2 \sin\varphi)^{k_1} (x_2 \cos\varphi + x_1 \sin\varphi)^{k_2}\ x_3$$

$=$ sum of monomials $\times$ coefficients.

Moreover, $e^{-|(x_1 \cos\varphi - x_2 \sin\varphi,\ x_2 \cos\varphi + x_1 \sin\varphi,\ x_3)|^2/2} = e^{|x|^2/2}$.

Essential self-adjointness now follows from the corollary to Stone's theorem.