Multiscale Substitutions

OR A Tail of Two Tiles

Yotam Smilansky, Manchester

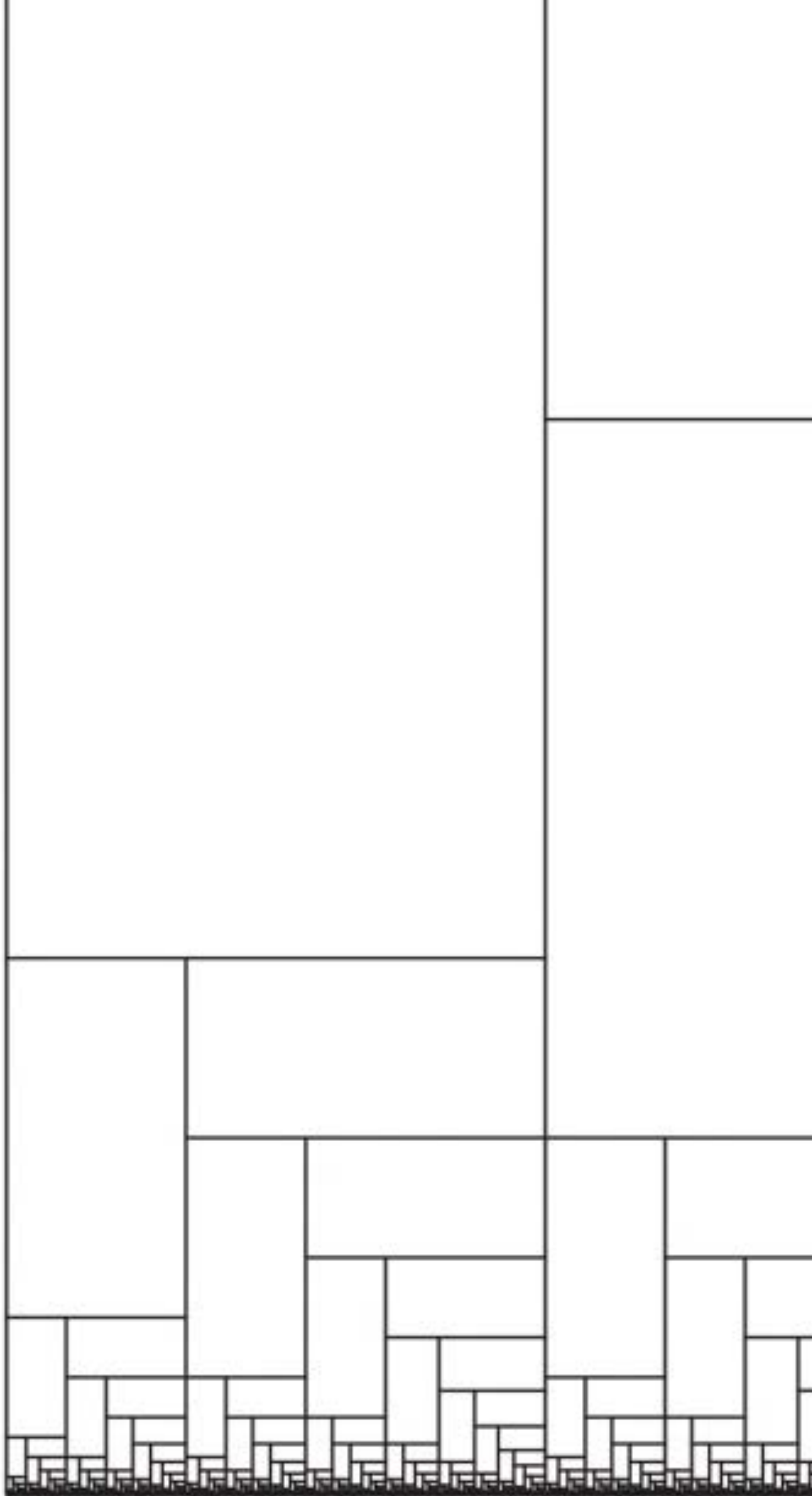
Aperiodic Order and Approximate Lattices II

Karlsruher Institut für Technologie, 2024

Partially based on joint work with Yaar Solomon

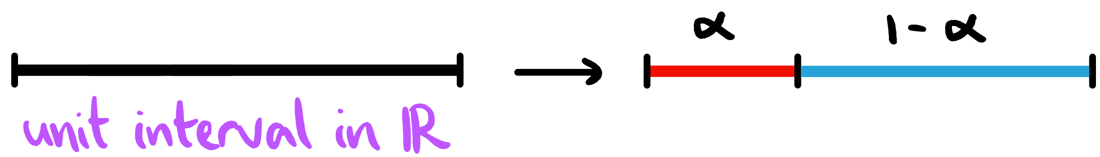
Plan of Talk

- Multiscale Substitutions
- Hyperbolic Tilings
- Statistics and Flows



Multiscale Substitutions

Let $0<\alpha<1$ and consider the substitution rule σ:

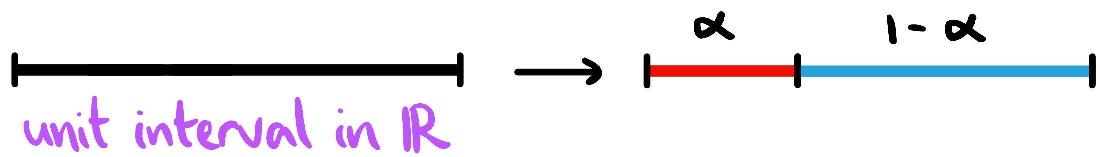


$$\frac{\log \frac{1}{\alpha}}{\log \frac{1}{1-\alpha}} \notin \mathbb{Q}$$

incommensurability

Multiscale Substitutions

Let $0<\alpha<1$ and consider the substitution rule σ:

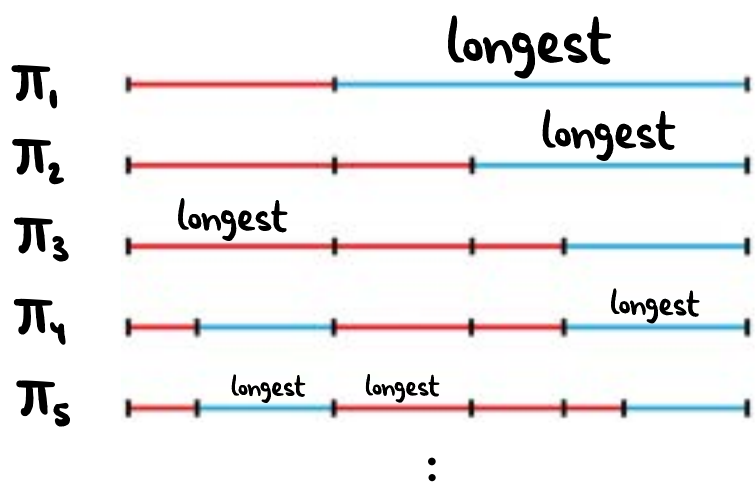


$$\frac{\log \frac{1}{\alpha}}{\log \frac{1}{1-\alpha}} \notin \mathbb{Q}$$

incommensurability

- sequences of partitions [Kakutani '76, S '20]

always split longest interval



$\Rightarrow$ uniform distribution, discrepancy, frequencies

Multiscale Substitutions

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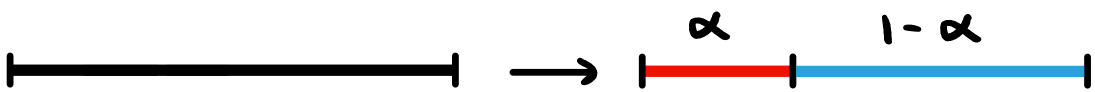


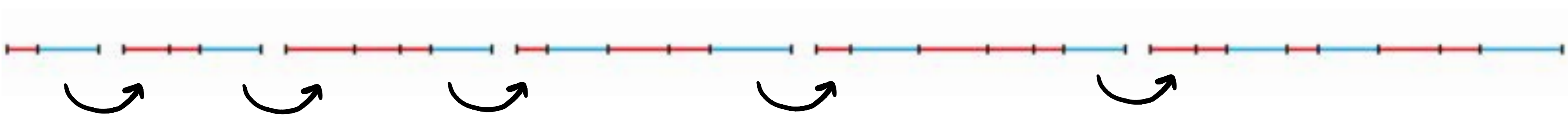
unit interval in ℝ

$$\frac{\log\frac{1}{\alpha}}{\log\frac{1}{1-\alpha}} \notin \mathbb{Q}$$

incommensurability

- multiscale substitution tilings [SS ´21, SS ´22, S ´22]

a time t dependent substitution semi-flow F_t : inflate by e^t and subsitute tiles of volume > 1



inflate with time, substitute largest tiles only

limit $\longrightarrow$

$\in X_\alpha$

ℝ-translations

$\Longrightarrow$ aperiodicity, repetitivity, unique ergodicity, gap distributions, non-BD

non-linear

Multiscale Substitutions

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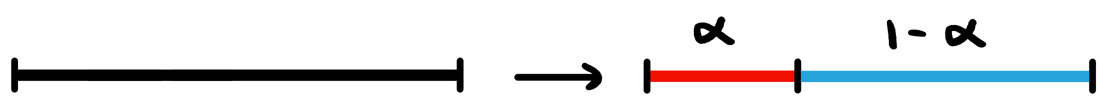


unit interval in $\mathbb{R}$

$$\frac{\log \frac{1}{\alpha}}{\log \frac{1}{1-\alpha}} \notin \mathbb{Q}$$

incommensurability

- α-words (temporary name...)

$\in X_\alpha$

projection $\downarrow$

$\cdots$ R B R R R B R R R B R R B R B R R B $\cdots$ $\in \Omega_\alpha$

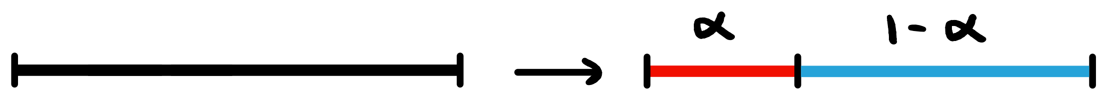
$\mathbb{Z}$ shift

symbolic dynamics in $\{R, B\}^{\mathbb{Z}}$

$\Longrightarrow$ aperiodicity, repetitivity, unique ergodicity, spectral properties?

Multiscale Substitutions

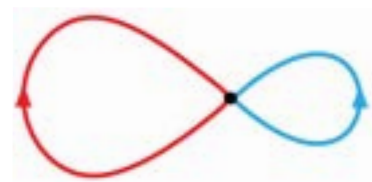
Let $0<\alpha<1$ and consider the substitution rule σ:



unit interval in $\mathbb{R}$

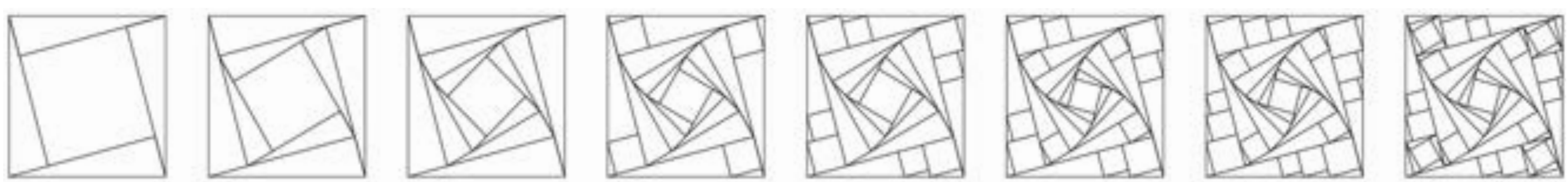
$$\frac{\log\frac{1}{\alpha}}{\log\frac{1}{1-\alpha}} \notin \mathbb{Q}$$

incommensurability

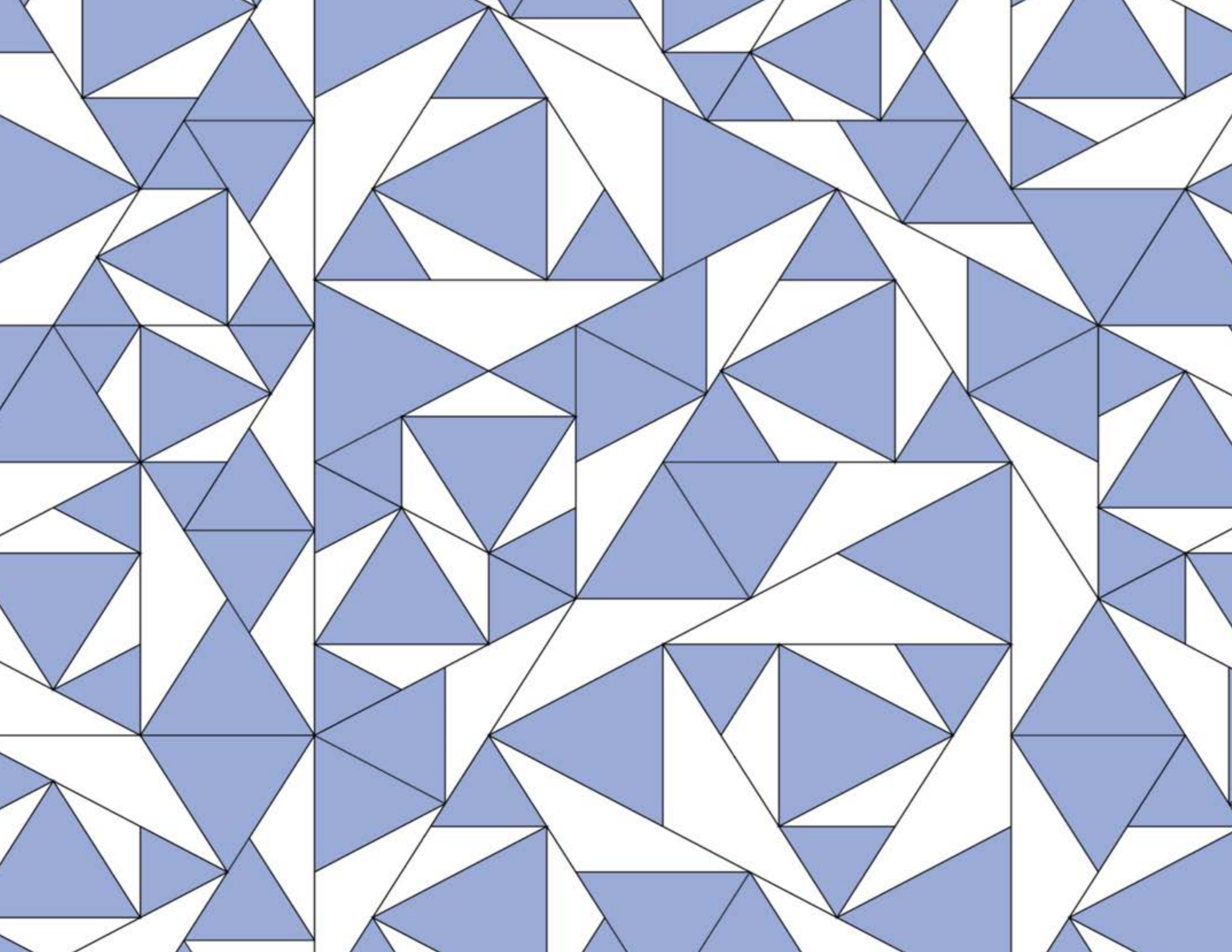
- associated graph G_σ [KSS '20] $\log\frac{1}{\alpha}$  $\log\frac{1}{1-\alpha}$

σ is incommensurable if G_σ contains two closed paths of lengths $\frac{a}{b} \notin \mathbb{Q}$

- higher dimensions and additional prototiles



Rauzy's fractal (commensurable). Does incommensurability rule out fractal boundaries?



Multiscale Substitutions

* most of these have since smashed

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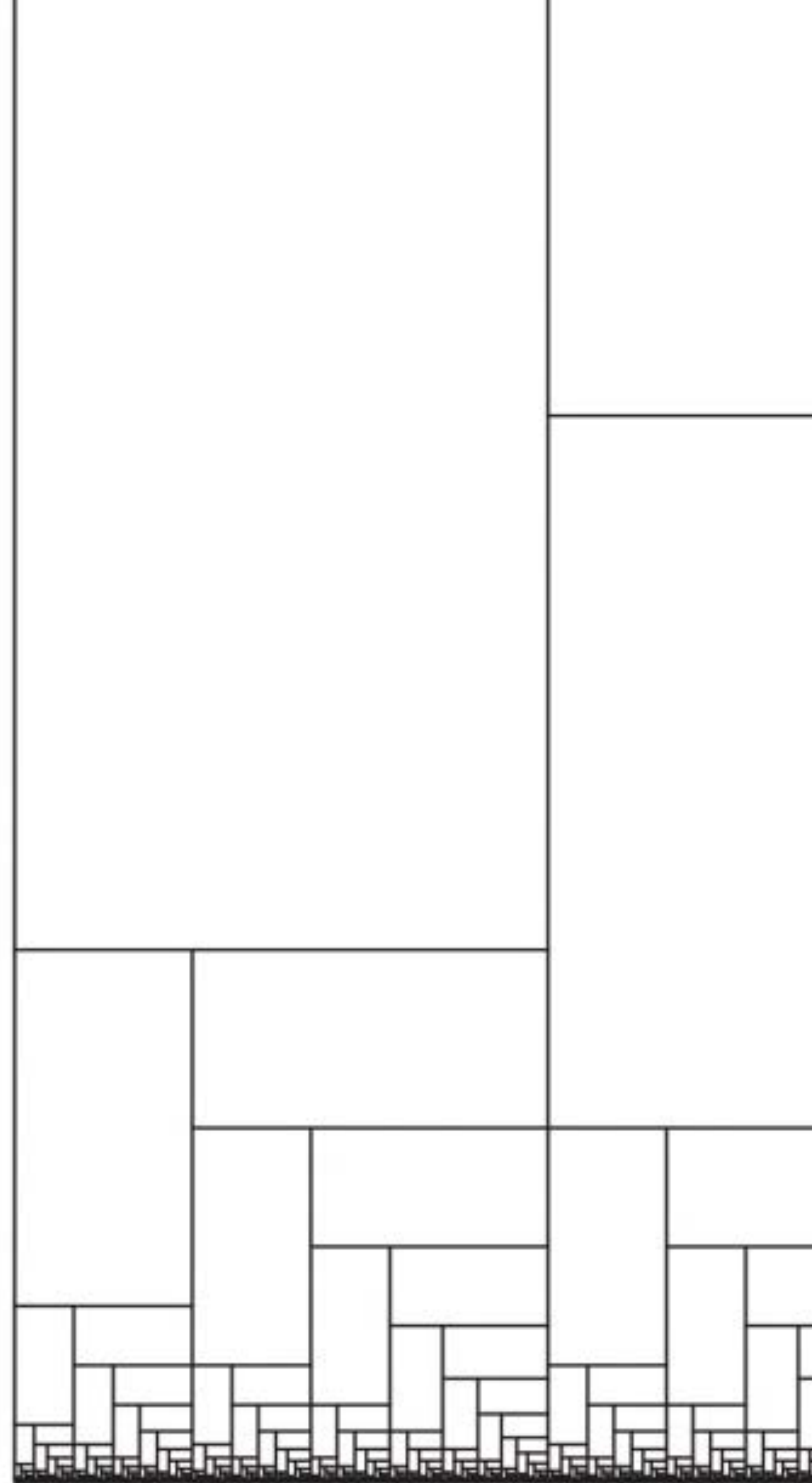
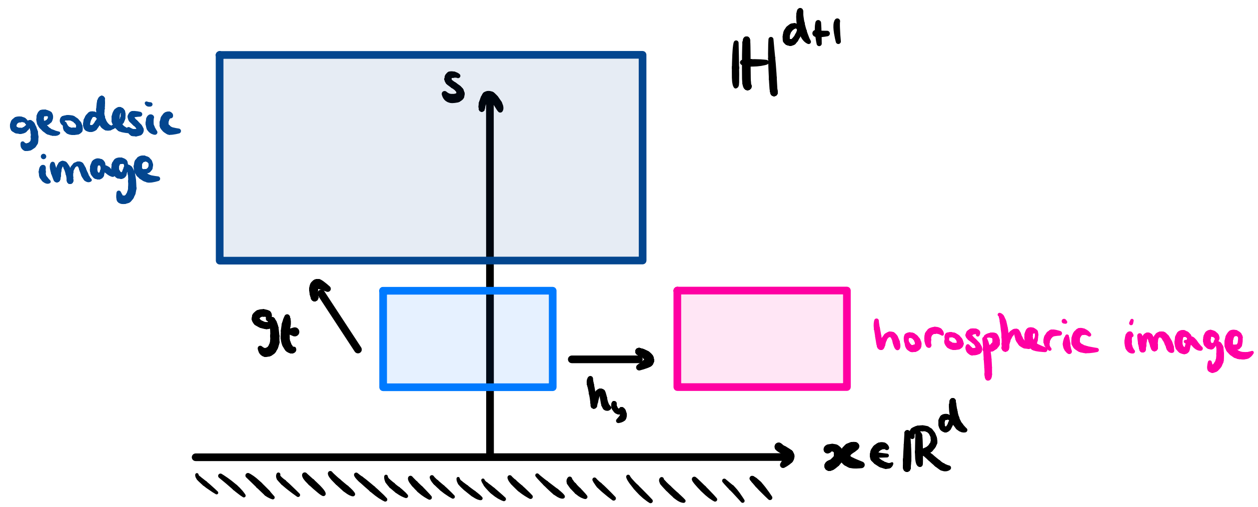


Hyperbolic Geometry

- upper half-space $\mathbb{H}^{d+1} = \{(x,s) : x = (x_1,\dots,x_d) \in \mathbb{R}^d,\ s>0\}$

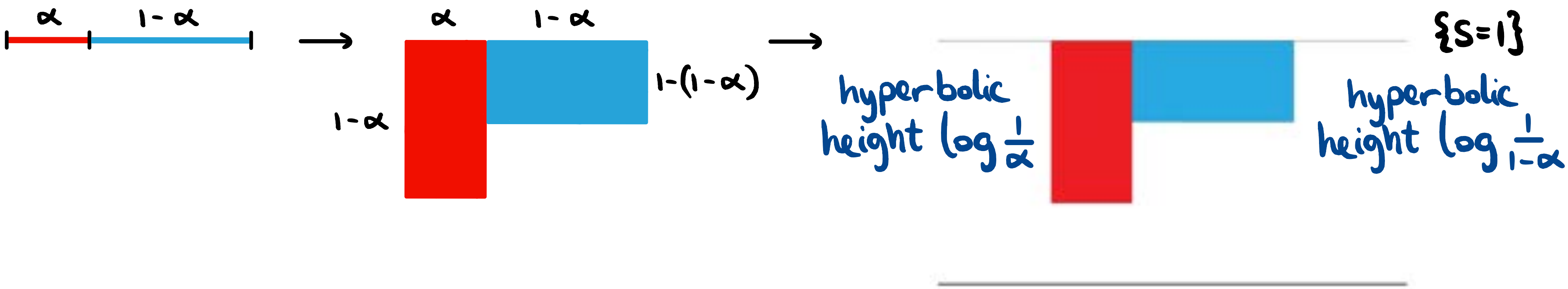


- two continuous actions by hyperbolic isometries:
  - horospheric $\mathbb{R}^d$-action $\quad h_y(x,s) = (y+x, s) \quad$ for $y \in \mathbb{R}^d$
  - geodesic $\mathbb{R}$-action $\quad g_t(x,s) = (e^t x, e^t s) \quad$ for $t \in \mathbb{R}$

satisfying $\quad g_t \circ h_y = h_{e^t y} \circ g_t$

From Substitutions to Hyperbolic Tilings

- give every tile a height corresponding to its scale, and position the patch in $\mathbb{H}^{d+1}$ aligned to the horosphere $\{s=1\}$



From Substitutions to Hyperbolic Tilings

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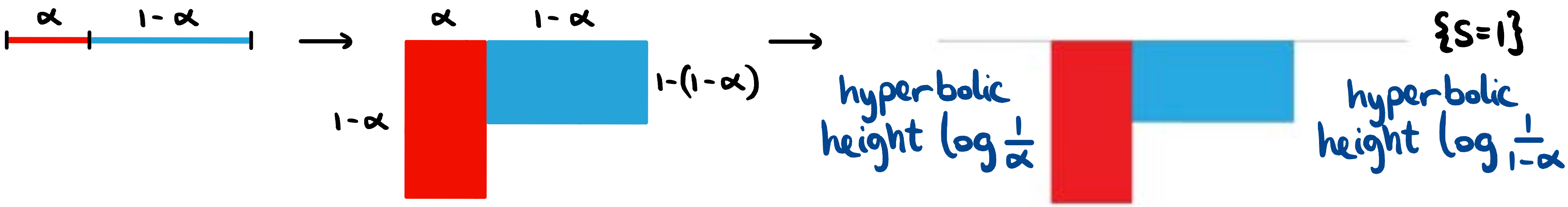
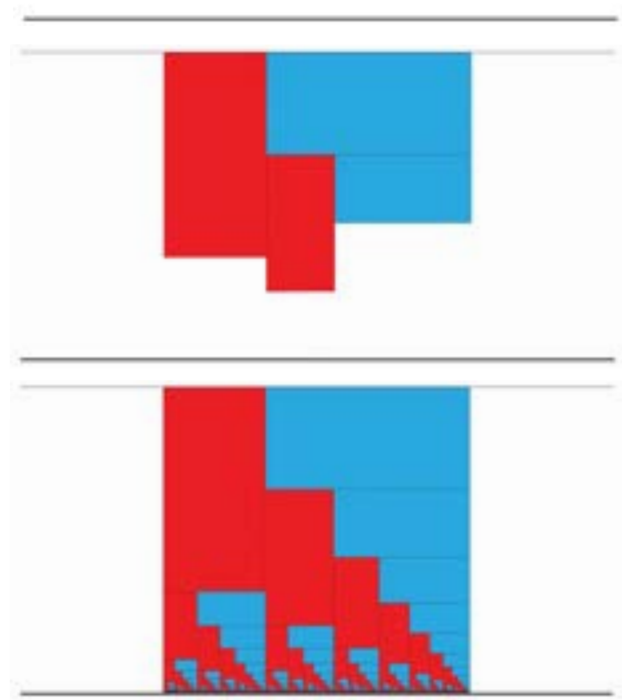


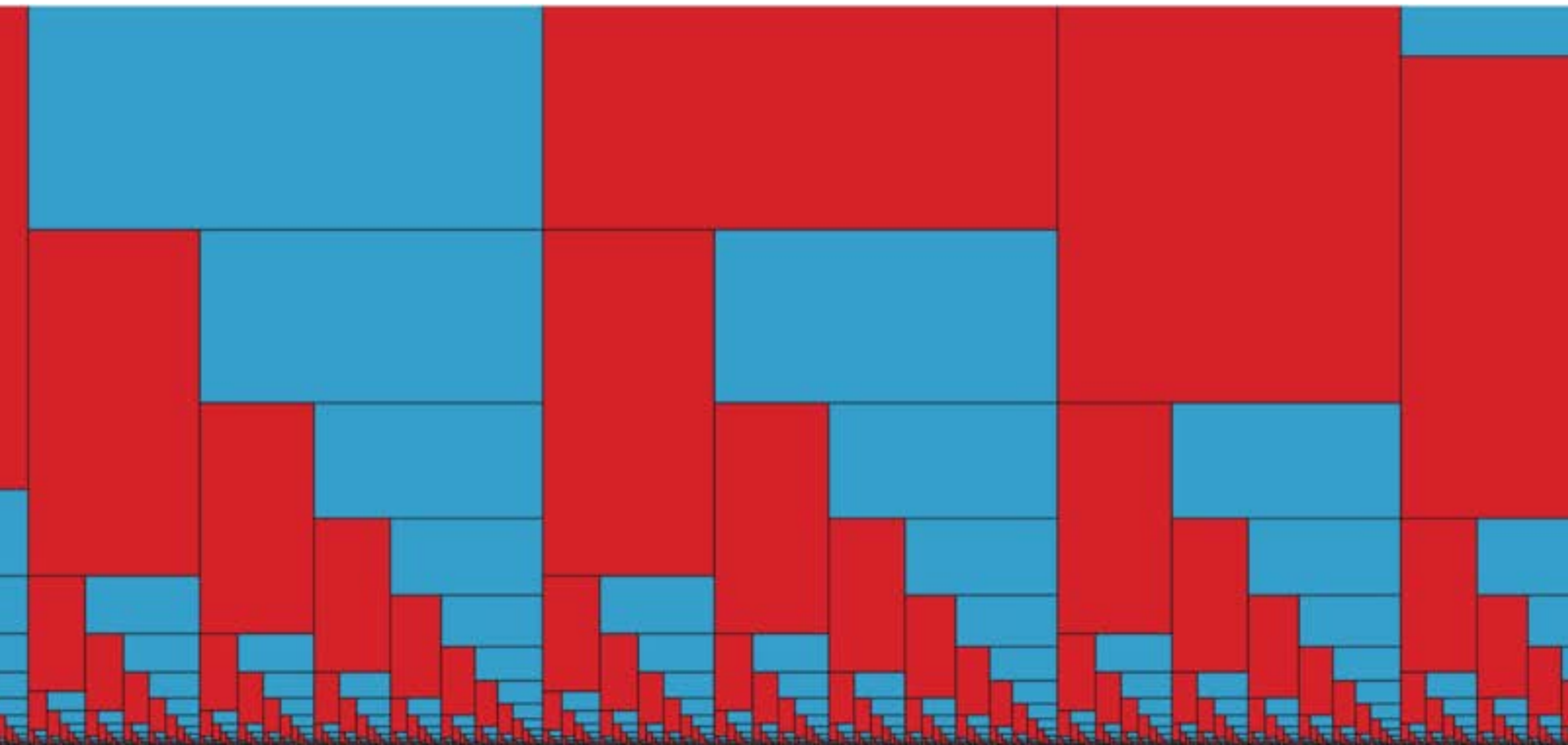
- glue an isometric copy of the patch to the bottom of a tile, repeat

- the limit object is called a tower



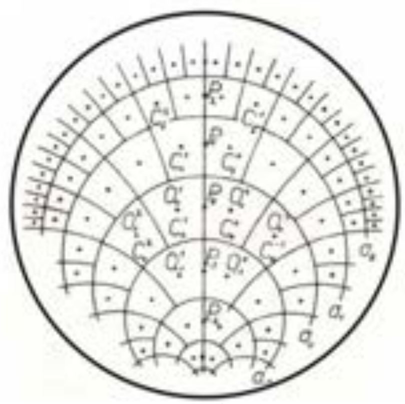
From Substitutions to Hyperbolic Tilings

- partial limits of towers under g_t as $t \to \infty$ define tilings of $\mathbb{H}^{d+1}$.
- the tiling space X_σ^{hyp} is the collection of all such limits.



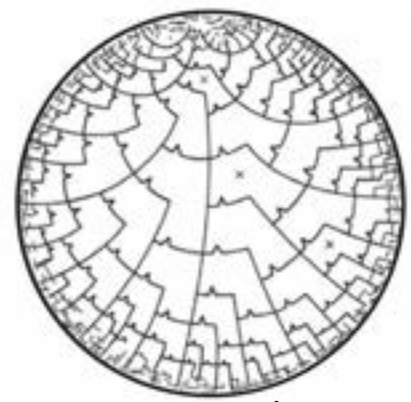
Related Hyperbolic Constructions

Poincaré disk model



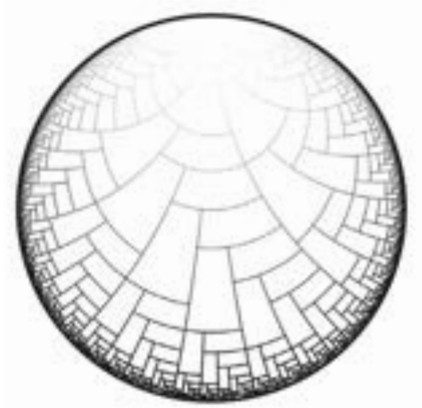
$\alpha = \frac{1}{2}$

[Böröczky '74][1]



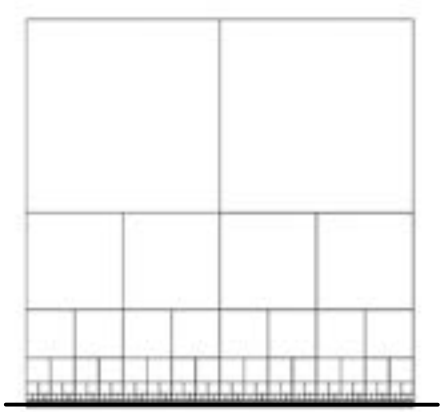
$\alpha = 1 - \frac{1}{\varphi}$

[Penrose '79][2]

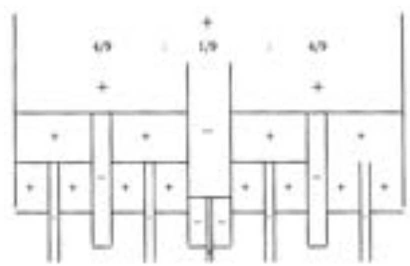


$\alpha = \frac{1}{3}$

Upper half-plane model

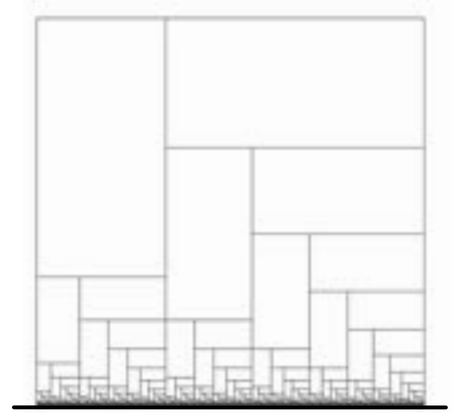


Binary tiling



Numeration systems

[Kamae '05][3]



$\alpha = \frac{1}{3}$

see also Escher's Regular Division of The Plane VI '57

1. Böröczky K., Gömbkitöltések állandó görbületű terekben I, Matematikai Lapok 25 (1974)
2. Penrose R., Pentaplexity: a class of nonperiodic tilings of the plane, Math. Intelligencer 2(1) (1979)
3. Kamae T., Numeration systems, fractals and stochastic processes, Israel J. Math. 149 (2005)

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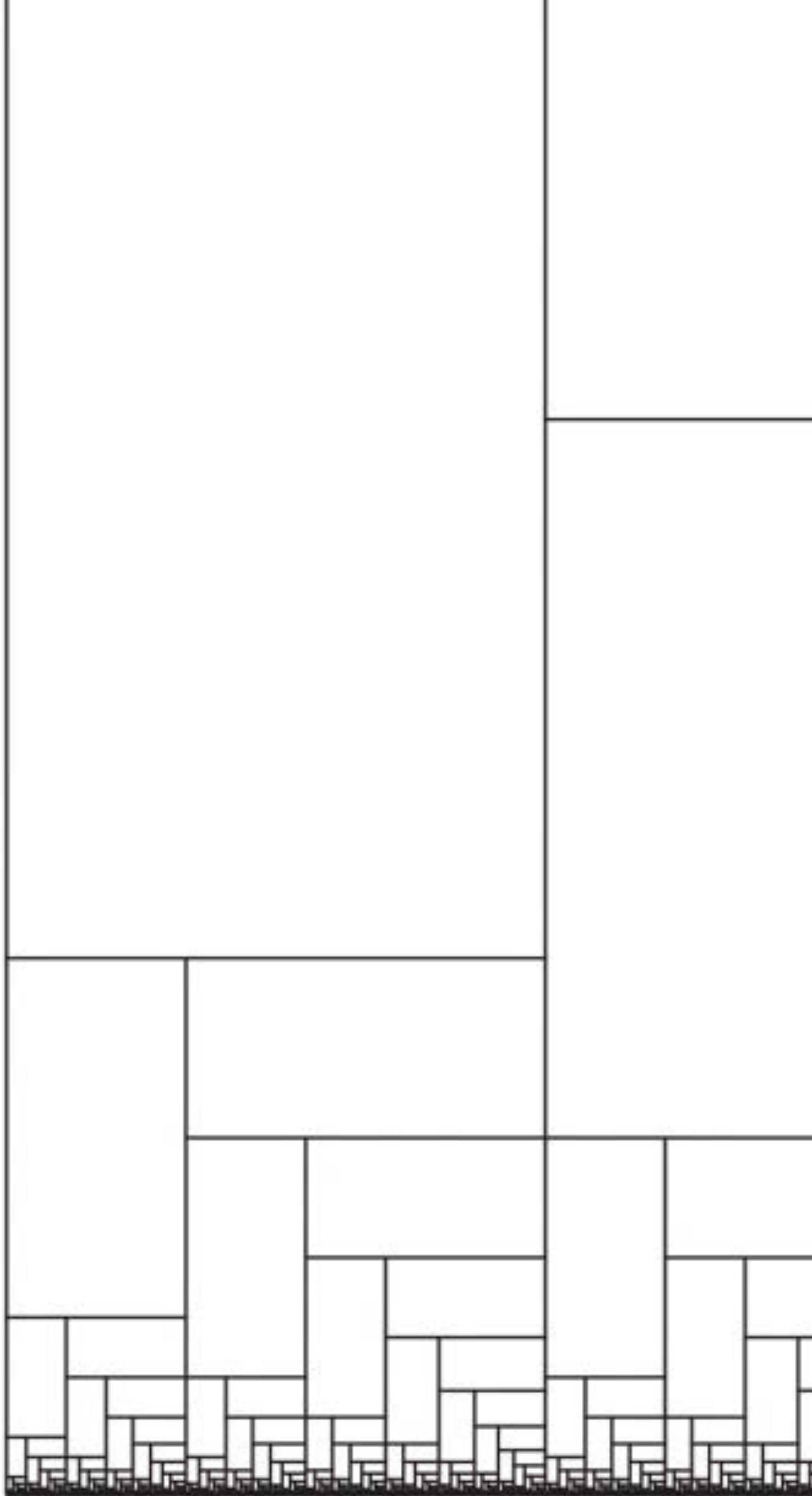
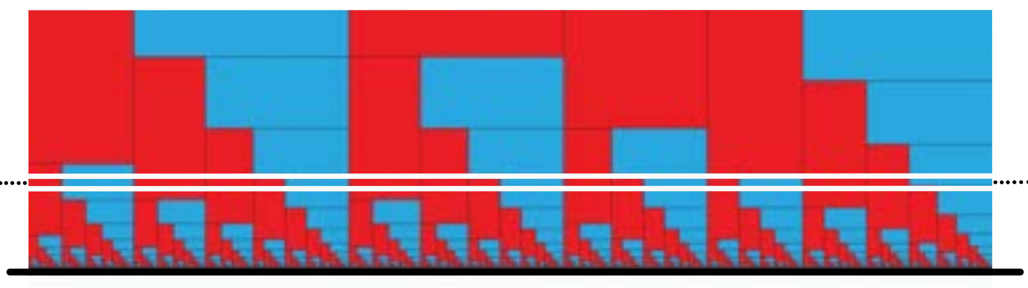


Liftings of Multiscale Substitution Tilings

Theorem Let $T \in X_\sigma^{hyp}$ be a $d+1$-dimensional hyperbolic tiling, then the map $T \mapsto T \cap \{s=1\}$ defines a d-dimensional multiscale substitution tiling generated by σ.



$\mathbb{H}^2$ $\qquad\qquad\qquad$ $\mathbb{R}$

$\mapsto$ 

$(\mapsto \cdots$ B R R R B R R R B R R B R B R R $\cdots)$

$\mathbb{Z}$

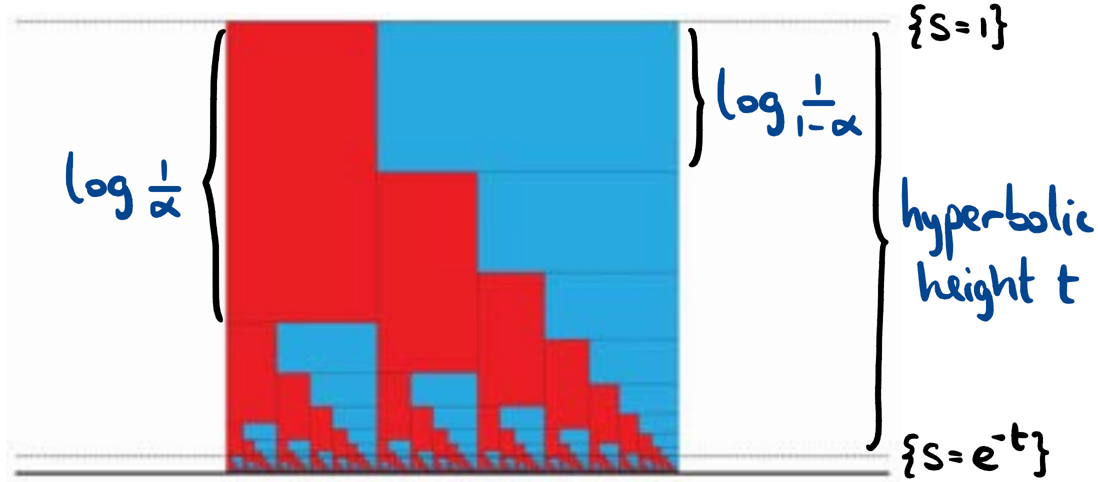
Finitely many tiles up to hyperbolic isometries

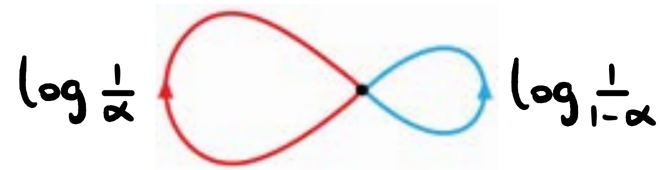
Infinitely many tiles up to Euclidean isometries (assuming incommensurability)

- horospheric $\mathbb{R}^d$-action h_y $\rightarrow$ translation action in $\mathbb{R}^d$ "space"
- geodesic $\mathbb{R}$-action g_t $\rightarrow$ substitution semiflow F_t "time"

Counting in Towers

Theorem Let σ be an incommensurable substitution in $\mathbb{R}^d$. Then:





$$\#\left\{\begin{array}{l}\text{tiles of type } j \text{ above}\\ \{s=e^{-t}\} \text{ inside a tower}\end{array}\right\} = \#\left\{\begin{array}{l}\text{Paths in } G_\sigma\\ \text{of length} < t\end{array}\right\} \sim \frac{[v^T \mathbb{1}]_j}{v^T H_\sigma \mathbb{1}} \cdot e^{dt}$$

$$\#\left\{\begin{array}{l}\text{tiles of type } j \text{ intersecting}\\ \{s=e^{-t}\} \text{ inside a tower}\end{array}\right\} = \#\left\{\begin{array}{l}\text{Walks in } G_\sigma\\ \text{of length} = t\end{array}\right\} \sim \frac{[v^T (S_\sigma - V_\sigma) \mathbb{1}]_j}{v^T H_\sigma \mathbb{1}} \cdot e^{dt}$$

combinatorics matrix $\displaystyle (S_\sigma)_{ij} = \sum_{\substack{T \text{ of type } j\\ \text{in } T_i}} 1$

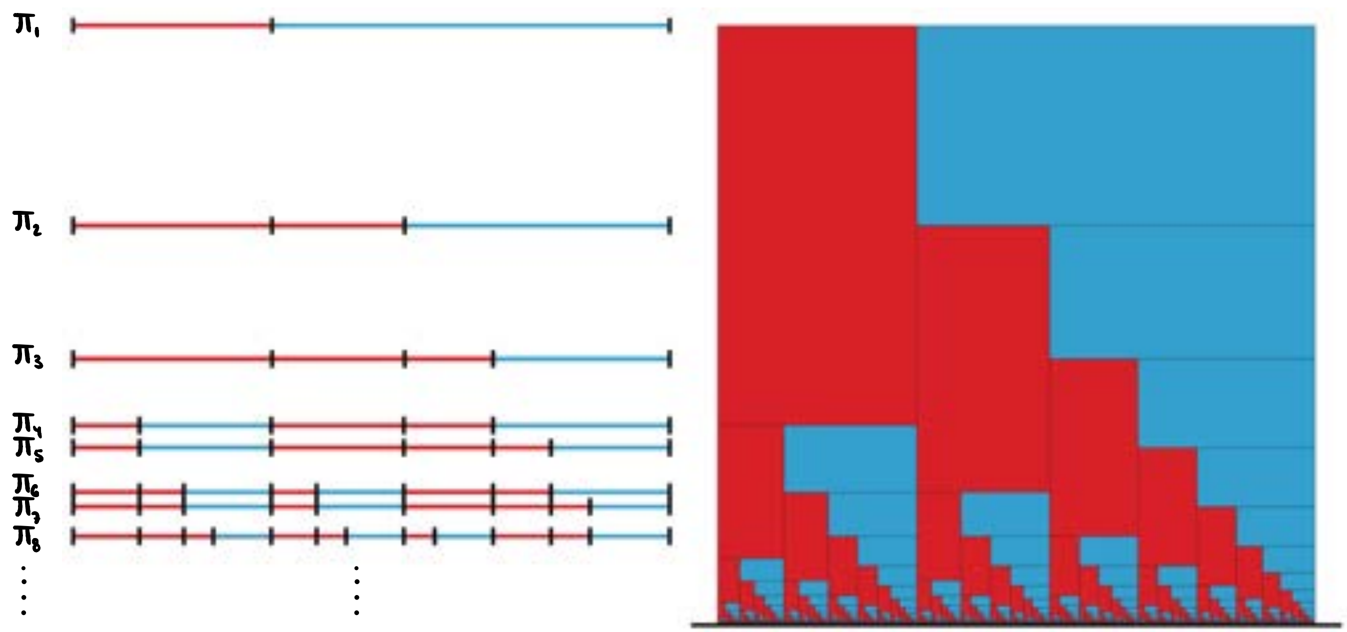
entropy matrix $\displaystyle (H_\sigma)_{ij} = \sum_{\substack{T \text{ of type } j\\ \text{in } T_i}} -\mathrm{vol}(T) \cdot \log \mathrm{vol}(T)$

volume matrix $\displaystyle (V_\sigma)_{ij} = \sum_{\substack{T \text{ of type } j\\ \text{in } T_i}} \mathrm{vol}(T)$

v^T = left Perron-Frobenius eigenvector of V_σ

Counting in Towers

Towers as suspended normalised Kakutani partitions



$$\lim_{m\to\infty} \frac{\#\text{ red intervals in } \pi_m}{\#\text{ intervals in } \pi_m} = \frac{2}{3} = \frac{\iint \frac{dx\,ds}{s^2}}{\iint \frac{dx\,ds}{s^2}}$$

$$\lim_{m\to\infty} \text{length(union of red intervals in } \pi_m) = \frac{-\frac{1}{3}\log\frac{1}{3}}{-\frac{1}{3}\log\frac{1}{3}-\frac{2}{3}\log\frac{2}{3}} = \frac{\iint \frac{dx\,ds}{s}}{\iint \frac{dx\,ds}{s}}$$

The Horospheric Flow

horospheric $\mathbb{R}^d$-action: $h_y(x,s) = (y+x,s)$ for $y \in \mathbb{R}^d$

Let σ be an incommensurable substitution in $\mathbb{R}^d$. Then

Theorem The dynamical system (X_σ^{hyp}, h_y) is minimal, that is, every orbit is dense.

Theorem Tilings in X_σ^{hyp} have no horospheric periods, that is, if $T \in X_\sigma^{hyp}$ and $y \in \mathbb{R}^d$ satisfy $h_y(T) = T$ then $y = 0$.

Proof Otherwise, $g_t(T) = g_t(h_y(T)) = h_{e^t y}(g_t(T)) \in X_\sigma^{hyp}$ has period $e^t y$ but is also arbitrarily close to a translation of T with period $y \neq e^t y$, which is impossible.

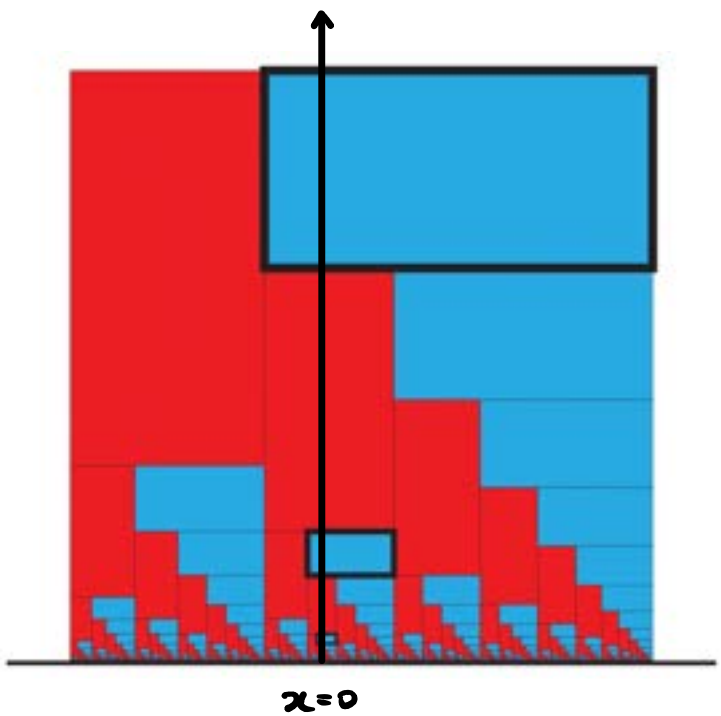
The Geodesic Flow

geodesic $\mathbb{R}$-action: $g_t(x,s) = (e^t x, e^t s)$ for $t \in \mathbb{R}$

Let σ be an incommensurable substitution in $\mathbb{R}^d$. Then

Theorem The dynamical system (X_σ^{hyp}, g_t) has dense orbits, periodic orbits (and orbits that are neither).

tiles intersecting the s-axis define words in the tile alphabet



periodic words $\Rightarrow$ periodic g_t orbits

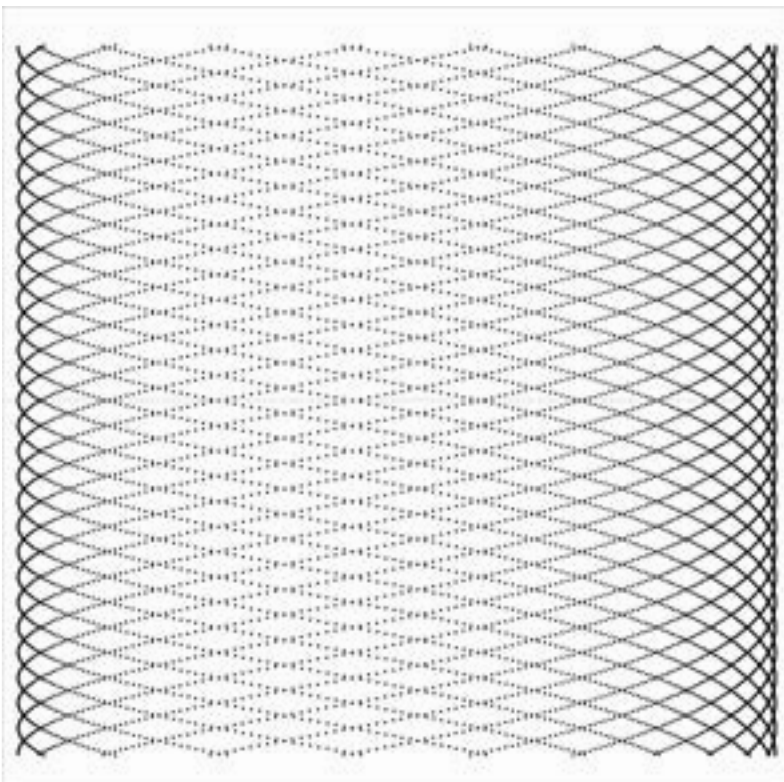
words containing every finite legal word $\Rightarrow$ dense g_t orbits

The Geodesic Flow

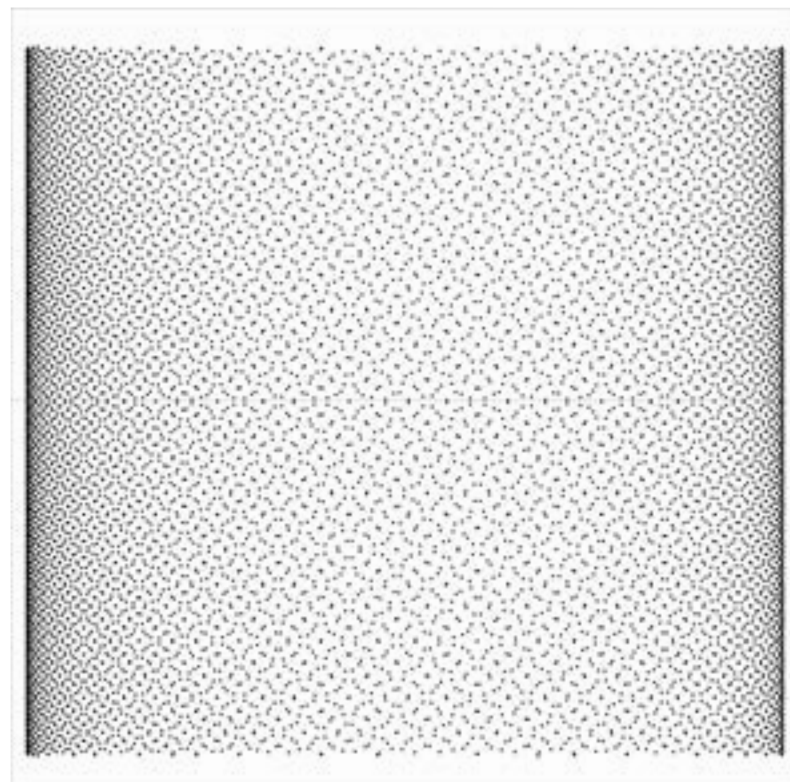
Theorem (Prime orbit theorem following [Parry, Pollicott '83])

$$\pi_\sigma(t) = \#\{\text{periodic orbits } \tau \text{ with minimal period } \lambda(\tau) \leq t\} \sim \frac{e^{dt}}{dt}, \quad t \to \infty$$

- Tiling zeta function $\zeta_\sigma(s) := \prod_\tau \left(1 - e^{-\lambda(\tau)s}\right)^{-1} \overset{(*)}{=} \frac{1}{\det(I - M_\sigma(s))} = \frac{1}{1 - \alpha^s - (1-\alpha)^s}$



$$\frac{\log \frac{1}{\alpha}}{\log \frac{1}{1-\alpha}} = \pi$$



$$\frac{\log \frac{1}{\alpha}}{\log \frac{1}{1-\alpha}} = \varphi$$

The Geodesic Flow

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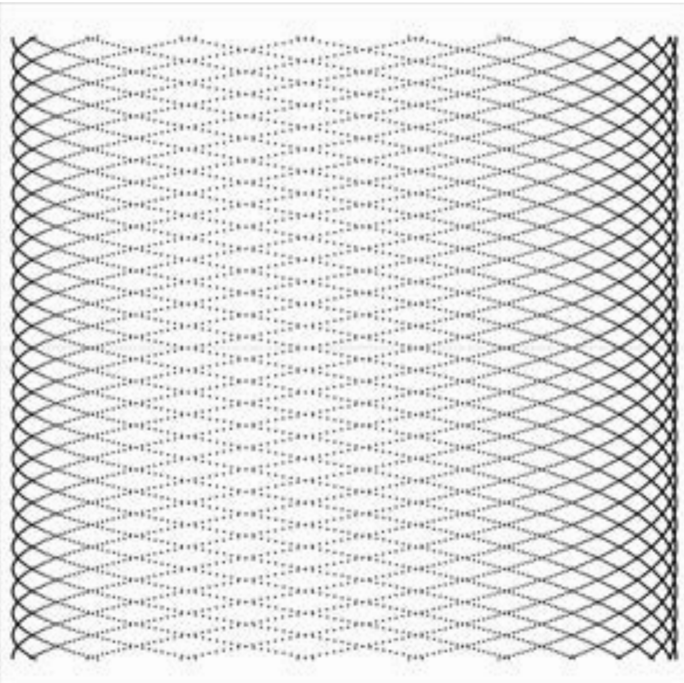
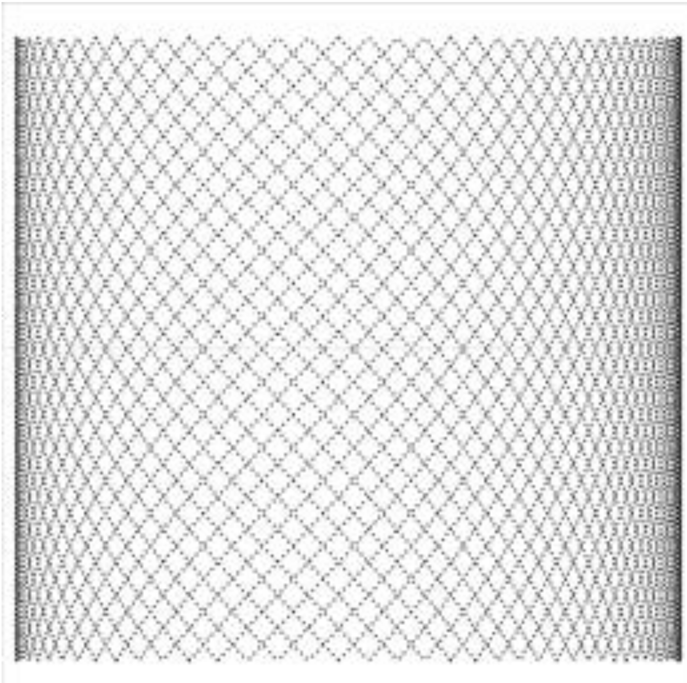
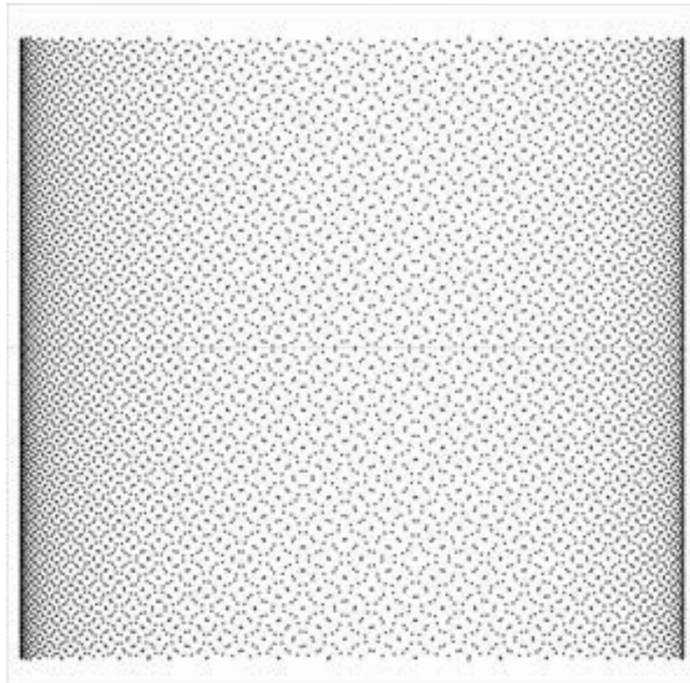


$\frac{\log \frac{1}{\alpha}}{\log \frac{1}{1-\alpha}} = \pi$



$\alpha = \frac{1}{3}$

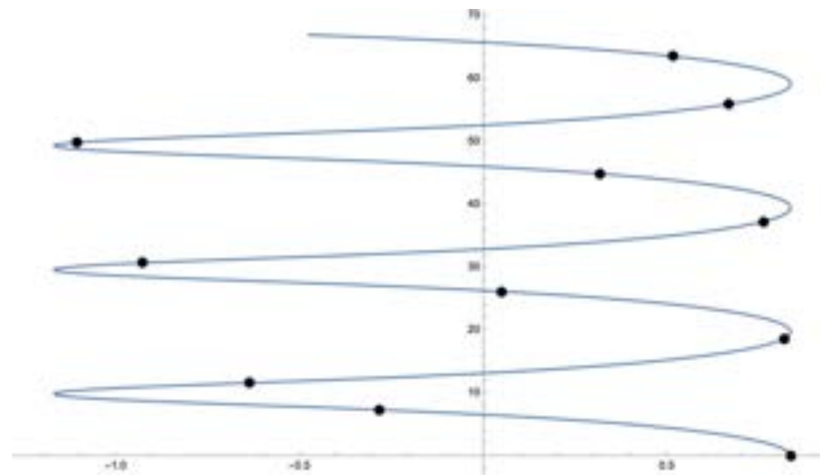


$\frac{\log \frac{1}{\alpha}}{\log \frac{1}{1-\alpha}} = \varphi$

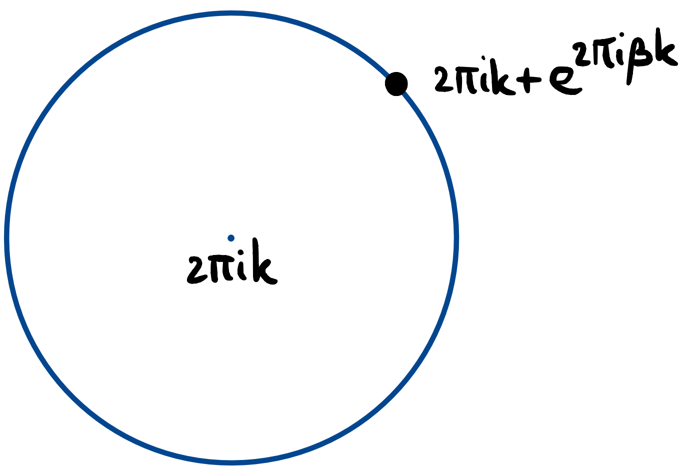
Zeroes of $1-\alpha^s-(1-\alpha)^s$ (w\ Alon Nishry, in progress)

A change of variables: $1-e^{\beta z}-e^{z}$, $\beta \notin \mathbb{Q}$, $0<\beta<1$

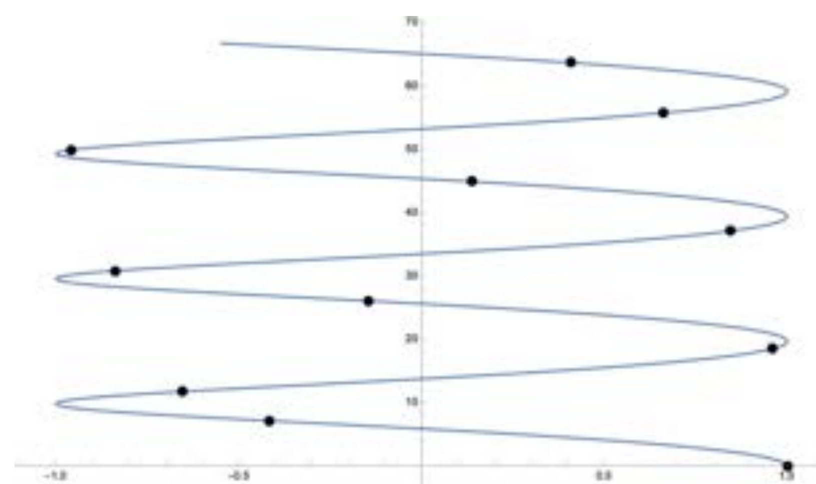
- Formula for the k'th zero z_k in terms of Fox-Wright functions.



- A "toy model" for the z_k's: $\{2\pi i k + \underbrace{e^{2\pi i \beta k}}_{\text{irrational rotation}} : k \in \mathbb{Z}\}$



$\Longrightarrow$



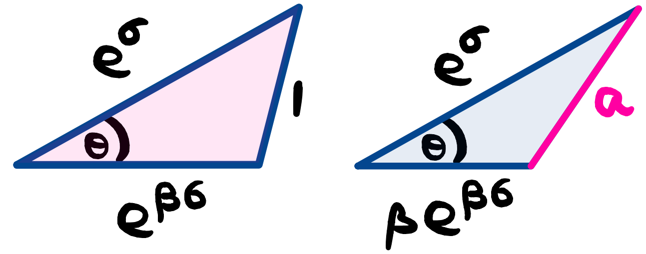
Zeroes of $1-\alpha^s-(1-\alpha)^s$ (w\ Alon Nishry, in progress)

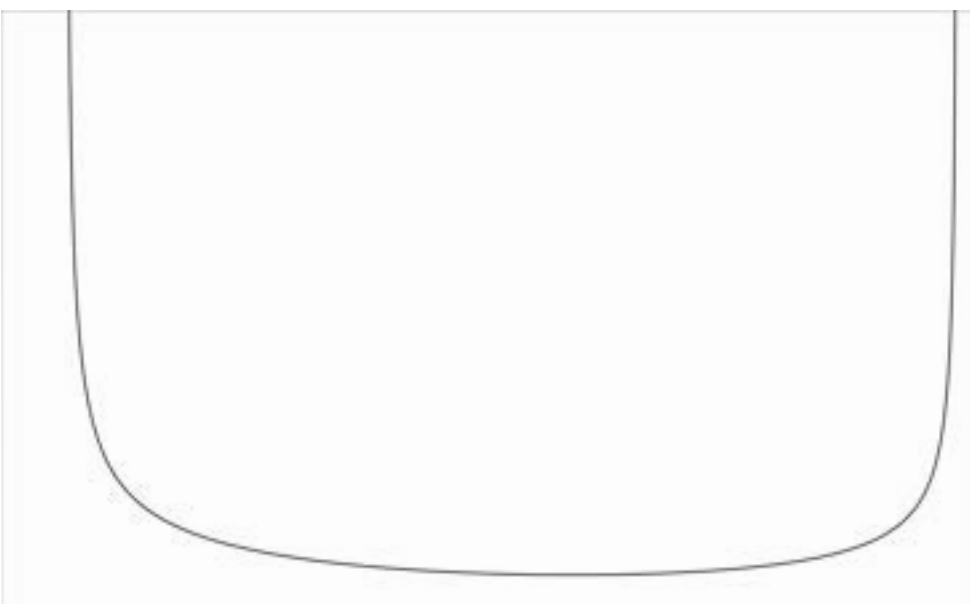
A change of variables : $1-e^{\beta z}-e^{z}$, $\beta\notin\mathbb{Q}$, $0<\beta<1$

- The distribution $G(\sigma)$ of real parts :

$$G(\sigma)\sim\frac{a^2}{\mathrm{Area}(\Delta\,1,e^{\sigma},e^{\beta\sigma})}$$

where





$G(\sigma)$ $\beta=1/\varphi$

Numerics $\beta=1/\varphi$

Zeroes of $1-\alpha^s-(1-\alpha)^s$ (w\ Alon Nishry, in progress)

A change of variables: $1-e^{\beta z}-e^{z}$, $\beta\notin\mathbb{Q}$, $0<\beta<1$

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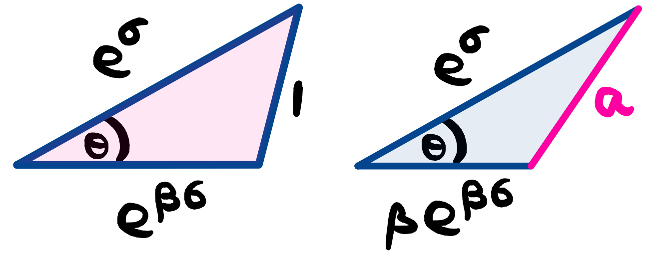
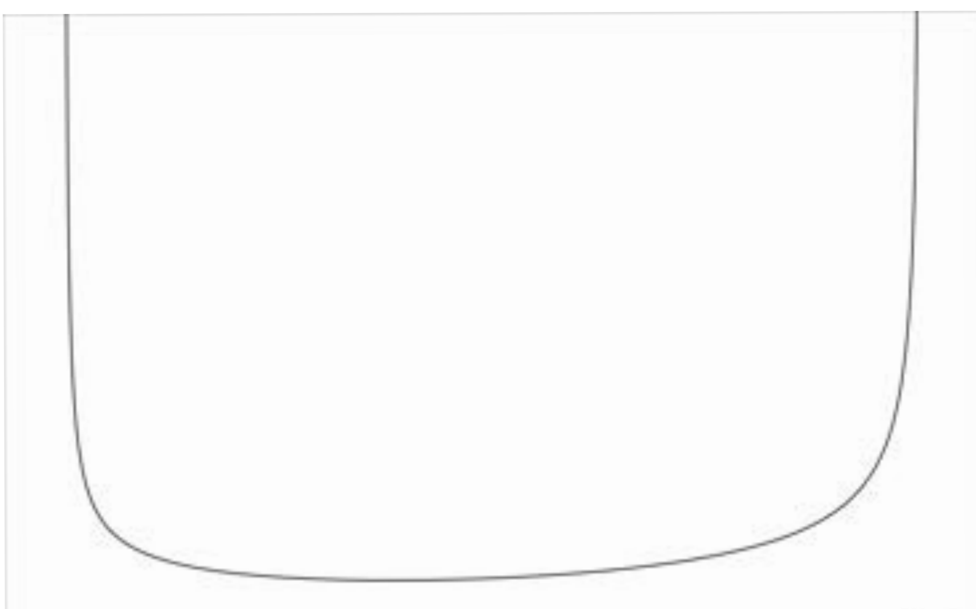
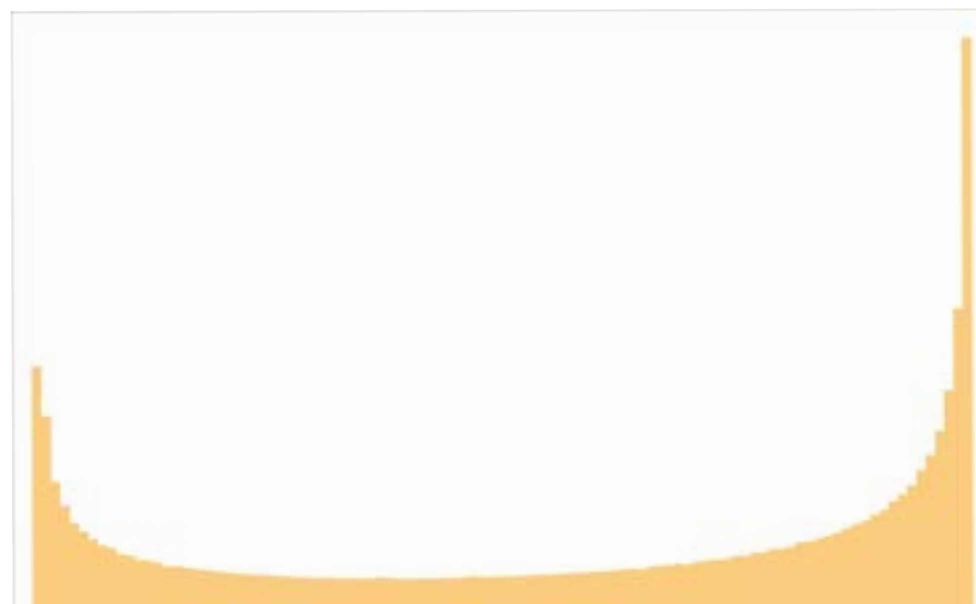




$G(\sigma)$ $\beta = 1/\pi$

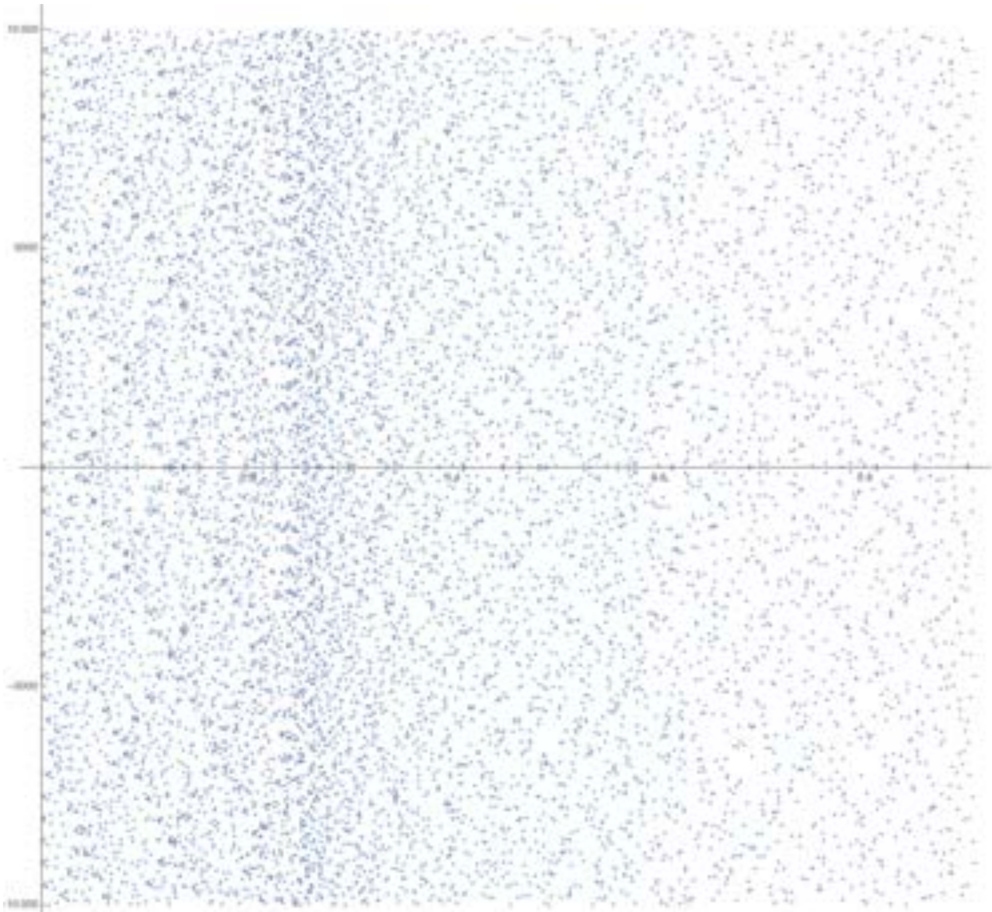


Numerics $\beta = 1/\pi$

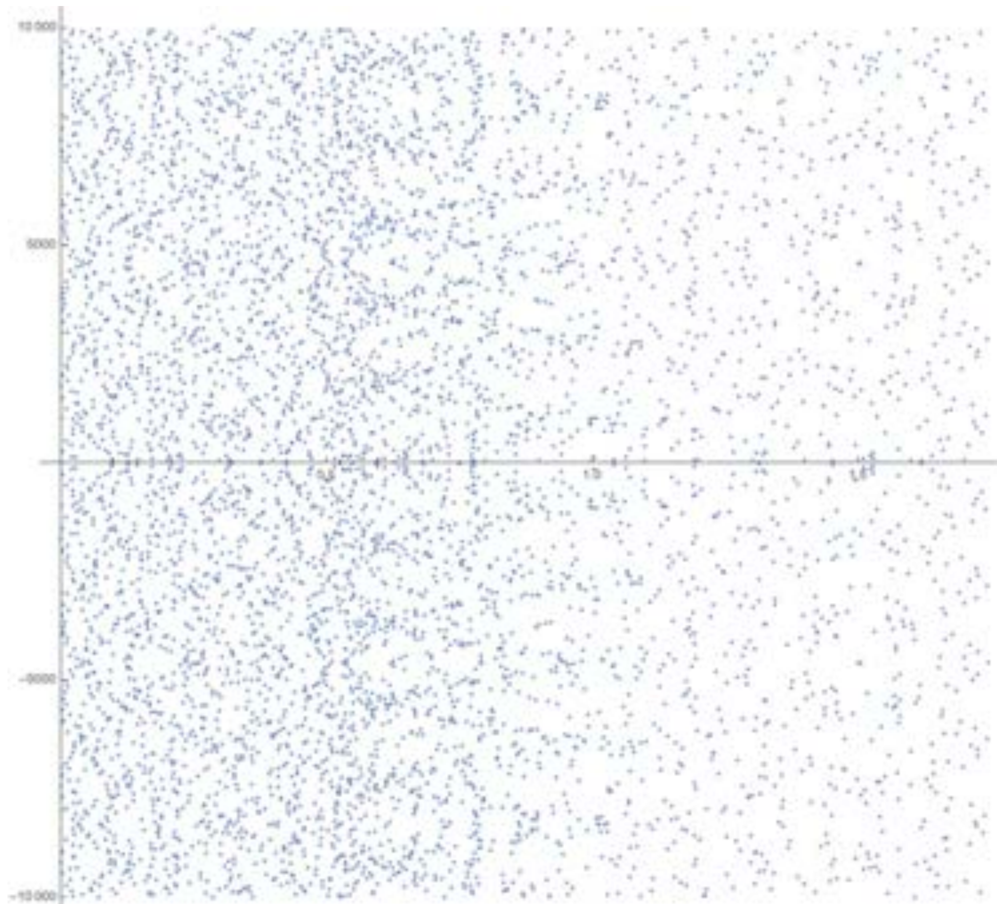
Zeroes of $1-\alpha^s-(1-\alpha)^s$ (w\ Alon Nishry, in progress)

The case $1-e^{\beta z}-e^{\gamma z}-e^{z}$, a "phase transition"?

"two generators" π and e



"two generators" $\sqrt{2}$ and $\sqrt{3}$



Thank You!