Lower bounds for multicolor Ramsey numbers

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Abstract

We give an exponential improvement to the lower bound on diagonal Ramsey numbers for any fixed number of colors greater than two.

1 Introduction

The Ramsey number $r(t;\ell)$ is the smallest natural number n such that every ℓ-coloring of the edges of the complete graph K_n contains a monochromatic K_t. For $\ell = 2$, the problem of determining $r(t) := r(t;2)$ is arguably one of the most famous in combinatorics. The bounds

$$\sqrt{2}^t < r(t) < 4^t$$

have been known since the 1940s, but, despite considerable interest, only lower order improvements [2, 6, 7] have been made to either bound. In particular, the lower bound $r(t) > (1+o(1))\frac{t}{\sqrt{2}e}\sqrt{2}^t$, proved by Erdős [3] as one of the earliest applications of the probabilistic method, has only been improved [7] by a factor of 2 in the intervening 70 years.

If we ignore lower order terms, the best known upper bound for $\ell \geq 3$ is $r(t;\ell) < \ell^{\ell t}$, proved through a simple modification of the Erdős–Szekeres neighborhood-chasing argument [4] that yields $r(t) < 4^t$. For $\ell = 3$, the best lower bound, $r(t;3) > \sqrt{3}^t$, again comes from the probabilistic method. For higher ℓ, the best lower bounds come from the simple observation of Lefmann [5] that

$$r(t;\ell_1+\ell_2) - 1 \geq (r(t;\ell_1)-1)(r(t;\ell_2)-1).$$

To see this, we blow-up an ℓ_1-coloring of $K_{r(t;\ell_1)-1}$ with no monochromatic K_t so that each vertex set has order $r(t;\ell_2)-1$ and then color each of these copies of $K_{r(t;\ell_2)-1}$ separately with the remaining ℓ_2 colors so that there is again no monochromatic K_t. By using the bounds $r(t;2)-1 \geq 2^{t/2}$ and $r(t;3)-1 \geq 3^{t/2}$, we can repeatedly apply this observation to conclude that

$$r(t;3k) > 3^{kt/2}, \qquad r(t;3k+1) > 2^t 3^{(k-1)t/2}, \qquad r(t;3k+2) > 2^{t/2}3^{kt/2}.$$

Our main result is an exponential improvement to all these lower bounds for three or more colors.

Our principal contribution is the following theorem, proved via a construction which is partly deterministic and partly random. The deterministic part shares some characteristics with a construction of Alon and Krivelevich [1], in that we consider a graph whose vertices are vectors over a finite field where adjacency is determined by the value of their scalar product, while randomness comes in through both random coloring and random sampling.

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Theorem 1. *For any prime q, $r(t;q+1) > 2^{t/2}q^{3t/8+o(t)}$.*

In particular, the cases $q = 2$ and $q = 3$ yield exponential improvements over the previous bounds for $r(t;3)$ and $r(t;4)$, both of which came from the probabilistic method (in fact, Lefmann's observation gives an additional polynomial factor in the four-color case, but this is of lower order than the exponential improvements that are our concern).

Corollary 2. $r(t;3) > 2^{7t/8+o(t)}$ *and* $r(t;4) > 2^{t/2}3^{3t/8+o(t)}$.

For the sake of comparison, we note that the improvement for three colors is from 1.732^t to 1.834^t, while, for four colors, it is from 2^t to 2.135^t. Improvements for all $\ell \geq 5$ now follow from repeated applications of Lefmann's observation, yielding

$$r(t;3k) > 2^{7kt/8+o(t)}, \qquad r(t;3k+1) > 2^{7(k-1)t/8+t/2}3^{3t/8+o(t)}, \qquad r(t;3k+2) > 2^{7kt/8+t/2+o(t)},$$

where we used, for instance,

$$r(t;3k+1)-1 \geq (r(t;3(k-1))-1)(r(t;4)-1) \geq (r(t;3)-1)^{k-1}(r(t;4)-1).$$

2 Proof of Theorem 1

Let q be a prime. Suppose $t \neq 0 \bmod q$ and let $V \subseteq \mathbb{F}_q^t$ be the set consisting of all vectors $v \in \mathbb{F}_q^t$ for which $\sum_{i=1}^t v_i^2 = 0 \bmod q$, noting that $q^{t-2} \leq |V| \leq q^t$. Here the lower bound follows from observing that we may pick $v_1, \dots, v_{t-2}$ arbitrarily and, since every element in $\mathbb{F}_q$ can be written as the sum of two squares, there must then exist at least one choice of v_{t-1} and v_t such that $v_{t-1}^2 + v_t^2 = -\sum_{i=1}^{t-2} v_i^2$.

We will first color all the pairs $\binom{V}{2}$ and then define a coloring of $E(K_n)$ by restricting our attention to a random sample of n vertices in V. Formally:

Coloring all pairs in $\binom{V}{2}$. For every pair $uv \in \binom{V}{2}$, we define its color $\chi(uv)$ according to the following rules:

- If $u \cdot v = i \bmod q$ and $i \neq 0$, then set $\chi(uv) = i$.
- Otherwise, choose $\chi(uv) \in \{q, q+1\}$ uniformly at random, independently of all other pairs.

Mapping $[n]$ into V. Take a random injective map $f : [n] \to V$ and define the color of every edge ij as $\chi(f(i)f(j))$.

Our goal is to upper bound the orders of the cliques in each color class.

Colors $1 \leq i \leq q-1$. Note that one cannot find an i-monochromatic clique of order larger than t for any $1 \leq i \leq q-1$. Indeed, suppose that $v_1, \dots, v_s$ form an i-monochromatic clique. We wish to show that they must be linearly independent and, therefore, that there are at most t of them. To this end, suppose that

$$u := \sum_{j=1}^{s} \alpha_j v_j = \bar{0}$$

and we wish to show that $\alpha_j = 0 \bmod q$ for all j. Observe that since $v_j \cdot v_j = 0 \bmod q$ for all j (our ground set V consists only of such vectors) and $v_k \cdot v_j = i \bmod q$ for each $k \neq j$, by considering all the products $u \cdot v_j$, we obtain that the vector $\bar{\alpha} = (\alpha_1, \ldots, \alpha_s)$ is a solution to

$$M\bar{\alpha} = \bar{0}$$

with $M = iJ - iI$, where J is the $s \times s$ all 1 matrix and I is the $s \times s$ identity matrix. In particular, we obtain that the eigenvalues of M (over $\mathbb{Z}$) are $is - i$ with multiplicity 1 and $-i$ with multiplicity $s-1$. Therefore, if $s \neq 1 \bmod q$, the matrix is also non-singular over $\mathbb{Z}_q$, implying that $\bar{\alpha} = 0$, as required. On the other hand, if $s = 1 \bmod q$, we can apply the same argument with $v_1, \ldots, v_{s-1}$ to conclude that $s - 1 \leq t$. But, we cannot have $s - 1 = t$, since this would imply that $t = 0 \bmod q$, contradicting our assumption. Therefore, we may also conclude that $s \leq t$ in this case.

Colors q and $q+1$. We call a subset $X \subseteq V$ a *potential clique* if $|X| = t$ and $u \cdot v = 0 \bmod q$ for all $u, v \in X$. Given a potential clique X, we let M_X be the $t \times t$ matrix whose rows consist of all the vectors in X. Observe that $M_X \cdot M_X^T = 0$, where we use the fact that each vector is self-orthogonal. First we wish to count the number of potential cliques and later we will calculate the expected number of cliques that survive after we color randomly and restrict to a random subset of order n.

Suppose that X is a potential clique and let $r := \mathrm{rank}(X)$ be the rank of the vectors in this clique, noting that $r \leq t/2$, since the dimension of any isotropic subspace of $\mathbb{F}_q^t$ is at most $t/2$. By assuming that the first r elements are linearly independent, the number of ways to build a potential clique X of rank r is upper bounded by

$$\left(\prod_{i=0}^{r-1} q^{t-i}\right) \cdot q^{(t-r)r} = q^{tr - \binom{r}{2} + tr - r^2} = q^{2tr - \frac{3r^2}{2} + \frac{r}{2}}.$$

This function is increasing up to $r = \frac{2t}{3} + \frac{1}{6}$, so the maximum occurs at $t/2$. By plugging this into the estimate above and summing over all possible ranks, we see that the number N_t of potential cliques in V is upper bounded by $q^{\frac{5t^2}{8} + o(t^2)}$.

The probability that a potential clique becomes monochromatic after the random coloring is $2^{1-\binom{t}{2}}$. Suppose now that p is such that $p|V| = 2n$ and observe that $p = nq^{-t+O(1)}$. If we choose a random subset of V by picking each $v \in V$ independently with probability p, the expected number of monochromatic potential cliques in this subset is, for $n = 2^{t/2} q^{3t/8 + o(t)}$,

$$p^t 2^{1-\binom{t}{2}} N_t \leq q^{-t^2 + o(t^2)} n^t 2^{-\frac{t^2}{2} + o(t^2)} q^{\frac{5t^2}{8} + o(t^2)} = \left(2^{-\frac{t}{2}} q^{-\frac{3t}{8} + o(t)} n\right)^t < 1/2.$$

Since our random subset will also contain more than n elements with probability at least $1/2$, there exists a choice of coloring and a choice of subset of order n such that there is no monochromatic potential clique in this subset. This completes the proof.

Remark. Our method also gives a new construction which matches Erdős' bound $r(t) > \sqrt{2}^t$ up to lower order terms. To see this, we set $V = \mathbb{F}_2^{2t}$ and color edges red or blue depending on whether $u \cdot v = 0$ or $1 \bmod q$. If we then sample $2^{t/2 + o(t)}$ vertices of V at random, we can show that w.h.p. the resulting set does not contain a monochromatic clique of order t. It was pointed out to us by Jacob Fox that one can achieve the same end by starting with any pseudorandom graph on n vertices for which the count of cliques and independent sets of order $2c\log_2 n$ is asymptotically the same as in $G(n,1/2)$ and sampling n^c vertices. This can be applied, for instance, with the Paley graph.

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