Supplementary problems on limits

1. Prove directly from the definition of a limit that

$$\lim_{n\to\infty} 2 + \frac{3}{n^2} = 2,$$

$$\lim_{n\to\infty} \frac{1}{n}\sin(n) = 0.$$

2. Prove from the definition of a limit that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} e^{\frac{n\pi i}{4}} = 0$$

and that

$$\lim_{n\to\infty} e^{\frac{n\pi i}{4}}$$

does not exist. (Here, $i = \sqrt{-1}$).

3. Prove that

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{2^k} = 2.$$

4. The Fibonacci sequence $\{f_n\}$ is defined by the relation $f_{n+2} = f_n + f_{n+1}$ and the initial values $f_0 = f_1 = 1$. Consider the sequence of ratios $r_n = \frac{f_{n+1}}{f_n}$. The goal of this exercise is to prove $\lim_{n\to\infty} r_n = \phi$, the so-called golden ratio, equal to $\frac{1+\sqrt{5}}{2} \approx 1.61$. The following is a suggestion how to proceed:

(0) Prove that if a_n is a sequence whose even and odd terms both converge to a limit L, then a_n also converges to the limit L.

(1) Deduce that $r_{n+1} = 1 + \frac{1}{r_n}$.

(2) Using the formula from (1), deduce by induction that the even and odd terms in the sequence (r_n) are monotone (in opposite directions!) and both bounded between 1 and 2.

(3) Using the fact that a bounded monotone sequence converges, deduce from the formula (1) that the even and odd terms in the sequence (r_n) both converge to ϕ. Apply (0) to deduce that (r_n) must also converge to ϕ.