p-adic Origamis

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ABSTRACT. An origami is a finite covering of a torus which is ramified over only one point. Origamis can be defined algebraically over an arbitrary field. In these notes, after a short reminder of complex origamis, we focus on origamis over p-adic fields with special emphasis on those that can be represented by Mumford curves. These *p-adic origamis*, at least those which are normal coverings of the torus, have been classified by K. Kremer. The main goal of this paper is to give a little background on p-adic uniformization and thus to introduce the reader to Kremer's work, the main results of which are summarized in the last section.

1. Introduction

These notes aim at giving an introduction to the interplay between complex and p-adic origamis for people who are familiar with Riemann surfaces, but not so much with their p-adic counterpart, the Mumford curves.

"Classical" complex origamis are a special class of translation surfaces, often also called "square-tiled surfaces". They have been studied a lot during the last 15 years from quite different points of view, e.g. by Gutkin-Judge [**6**], Lochak [**14**] (who coined the name "origami"), Schmithüsen [**18**], Hubert-Lelièvre [**11**], Eskin-Kontsevich-Zorich [**3**], Matheus-Möller-Yoccoz [**15**] and many others. They have a very simple combinatorial description by gluing squares, see Section 2, but nevertheless encode a very rich and deep structure. In particular, every origami determines an algebraic curve in moduli space, a so called *Teichmüller curve*. Each of these curves contains point representing Riemann surfaces which are (as complex algebraic curves) defined over a number field, and thus also determine a projective curve over the field $\mathbb{C}_p$, the completion of the algebraic closure of the field of p-adic numbers.

Over p-adic fields like $\mathbb{C}_p$ and $\mathbb{Q}_p$, there is a meaningful notion of analytic functions and of analytic spaces. But in contrast to the complex case there is no uniformization theorem for one-dimensional p-adic analytic manifolds. At least the easy part of the uniformization theorem holds: any quotient of an open subset of the projective line $\mathbb{P}^1$ by a discontinuously acting group of Möbius transformations is an analytic manifold. These are considered as the analogues of Riemann surfaces.

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If such a quotient is compact it carries, as in the complex case, the structure of a nonsingular projective algebraic curve. But unlike the classical complex situation, not every nonsingular projective algebraic curve over $\mathbb{C}_p$ can be obtained this way. An algebraic curve over $\mathbb{C}_p$ that can be represented as quotient of an open subset of $\mathbb{P}^1(\mathbb{C}_p)$ by a discontinuous subgroup of $\mathrm{PGL}_2(\mathbb{C}_p)$, is called a *Mumford curve.* We shall explain this concept from different points of view in Sections 4, 5 and 6.

We call an origami over $\mathbb{C}_p$ a *p-adic origami* if it is a covering of Mumford curves. The task of describing all p-adic origamis has been done to a large extent by Karsten Kremer in his PhD thesis [**12**] and the subsequent paper [**13**]: he succeeded in classifying all normal p-adic origamis. In Section 7, we shall discuss one of his two basic examples in some detail, and then sketch Kremer's general results.

2. Complex origamis

The most elementary definition of a complex origami is the following, which also explains the name "square-tiled surface":

Definition 2.1. An *origami* is a closed surface X that can be obtained from finitely many squares in the euclidean plane by gluing each left side of a square to a right side and each upper side to a lower side in such a way that all gluings are performed by translations in the plane.

On such a surface there is an obvious notion of horizontal (and vertical) cylinders. Numbering the squares by the integers $1, \dots, d$, the decomposition of X into horizontal (resp. vertical) cylinders corresponds to the cycle decomposition of a permutation σ_h (resp. σ_v) in the symmetric group S_d. That X is connected implies that the subgroup of S_d generated by σ_h and σ_v acts transitively on $\{1, \dots, d\}$. Conversely every such pair of permutations determines an origami.

A surface X as in Definition 2.1 comes along with a surjective map $p : X \to E$ to the torus $E = \mathbb{R}^2/\mathbb{Z}^2$: on each square of X, the map p identifies the left and the right edge, and the top and the bottom edge. Clearly this is compatible with the gluings that define X. Note that outside the vertices of the squares, p is an unramified covering.

Identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way, the torus $E = \mathbb{C}/(\mathbb{Z} \oplus \mathrm{i}\mathbb{Z})$ is not only a topological surface, but also a complex torus and a complex projective nonsingular curve of genus 1. Since translations are in particular holomorphic functions, the gluing rules in Definition 2.1 turn X into a compact Riemann surface and thus also a complex nonsingular projective curve. The map p described above is then a holomorphic map between Riemann surfaces, and a finite morphism between algebraic curves. This observation suggests a definition of origami over more general fields than the complex numbers:

Definition 2.2. Let k be an algebraically closed field. An *origami* over k is a finite morphism $p : X \to E$ between nonsingular projective curves X and E over k such that E has genus 1 and p is ramified over at most one point of E.

Remark 2.3. 1) To make sure that the morphism p is finite it suffices to require that it be nonconstant.
2) The genus of X is at least 1, and if it is 1 then p is unramified by the Riemann-Hurwitz formula (in this case p is an *isogeny*).
3) The assumption that k is algebraically closed can be dropped if an appropriate

notion of "curve over k" is used. In fact we shall be interested later in Riemann surfaces that "can be defined over a number field K", by which we mean that there is a a scheme X_0 over K from which X is obtained by base change from K to $\mathbb{C}$, i. e. $X = X_0 \otimes_K \mathbb{C}$.

3. Teichmüller curves

As indicated above, origamis are special translation surfaces. We now make this notion precise:

DEFINITION 3.1. A 2-dimensional manifold X with a given atlas of local charts is called a *strict translation surface* if all transition functions between the charts are translations of the euclidean plane.

Clearly, if X is an origami as in Definition 2.1 and $\Sigma \subset X$ is the finite set of vertices of the squares, then $X^* := X \setminus \Sigma$ is a strict translation surface. This observation motivates the following definition:

DEFINITION 3.2. A 2-dimensional manifold X is called a *translation surface* if there is a discrete subset $\Sigma \subset X$ such that $X^* := X \setminus \Sigma$ is a strict translation surface.

REMARK 3.3. Every translation surface is also a Riemann surface. This is obvious for strict translation surfaces since translations are holomorphic maps. The general case follows from the Riemann extension theorem.

REMARK 3.4. The differential dz in the complex plane pulls back via the translation atlas to a holomorphic differential on every translation surface.
Conversely, every translation surface X can be obtained from a Riemann surface X and the choice of a nonzero holomorphic differential ω on X: Let Σ be the set of zeroes of ω, and define charts on simply connected open subsets $U \subset X \setminus \Sigma$ by $P \mapsto \int_{P_0}^{P} \omega$ (for some $P_0 \in U$).

This remark shows that compact translation surfaces of genus $g \geq 1$ are classified by the space ΩM_g of pairs (S, ω), where S is a Riemann surface of genus g and ω is a nonzero holomorphic differential on S. Note that ΩM_g is a vector bundle of rank g on M_g from which the zero section is removed.
There is a natural action of $\mathrm{SL}_2(\mathbb{R})$ on the space ΩT_g of marked translation surfaces: given a matrix $A \in \mathrm{SL}_2(\mathbb{R})$ and a translation surface X, we obtain a new translation surface X_A by postcomposing all chart maps with the $\mathbb{R}$-linear map on $\mathbb{R}^2$ induced by A. If a point X in ΩT_g is considered as a pair (S, ω) with a marked Riemann surface S and a nonzero holomorphic 1-form ω on S, the point X_A corresponds to (S_A, ω_A), where $\omega_A = A \cdot \begin{pmatrix} \mathrm{Re}\, \omega \\ \mathrm{Im}\, \omega \end{pmatrix}$ and S_A is the unique complex structure on the surface S such that ω_A is holomorphic.
For given $X = (S, \omega) \in \Omega T_g$ this action induces a map from $\mathrm{SL}_2(\mathbb{R})$ to T_g by $A \mapsto S_A$. Note that S_A is the same marked Riemann surface as S if and only if $A \in \mathrm{SO}_2(\mathbb{R})$. Thus we obtain an embedding ι_X of $\mathbb{H} = \mathrm{SO}_2(\mathbb{R}) \backslash \mathrm{SL}_2(\mathbb{R})$ into T_g. Such embeddings are called *Teichmüller embeddings*, and the image $\Delta_X = \iota_X(\mathbb{H})$ in T_g is called a *Teichmüller disk*.

PROPOSITION 3.5. *Teichmüller embeddings are holomorphic and isometric (for the hyperbolic metric on $\mathbb{H}$ and the Teichmüller metric on T_g).*

Proof. A proof of this proposition can be found in [**9**], Thm. 3.4. □

Recall that the mapping class group Mod_g acts properly discontinuously and isometrically on T_g, and the quotient space $M_g = T_g/\mathrm{Mod}_g$ is the moduli space of Riemann surfaces of genus g, which is known to be a quasiprojective variety. We are interested in the images of Teichmüller disks in moduli space. In general, this is a weird (and dense) subset of M_g, but in certain cases it has a much nicer structure:

Proposition 3.6. *Let X be a translation surface of genus $g \geq 1$ and Δ_X the corresponding Teichmüller disk. Denote by $\widetilde{\Gamma}_X$ the stabilizer of Δ_X in Mod_g and by Γ_X the factor group of $\widetilde{\Gamma}_X$ by the (finite) pointwise stabilizer of Δ_X.*
Then Γ_X acts as a Fuchsian group on $\mathbb{H} = \Delta_X$, and if Γ_X is a lattice in $\mathrm{SL}_2(\mathbb{R})$ the following holds:

(i) *Δ_X/Γ_X is a Riemann surface of finite type.*
(ii) *The image C_X of Δ_X in M_g is birational to Δ_X/Γ_X.*

Moreover $\widetilde{\Gamma}_X$ is isomorphic to the group $\mathrm{Aff}^+(X)$ of orientation preserving affine diffeomorphisms of X, and Γ_X is isomorphic to the image of $\mathrm{Aff}^+(X)$ in $\mathrm{SL}_2(\mathbb{R})$, i. e. the group of linear parts of the affine diffeomorphisms.

Proof. For a proof of most of the claims in this proposition see [**9**]. □

The group Γ_X in the proposition is called the *Veech group* of the translation surface X. If Γ_X is a lattice in $\mathrm{SL}_2(\mathbb{R})$, the proposition implies that Δ_X/Γ_X and C_X are complex algebraic curves. In this case C_X, considered as a subvariety of the moduli space M_g, is called a *Teichmüller curve.* As already observed by Veech [**20**], Teichmüller curves are never projective, hence always have "cusps" at the boundary of M_g.
For an arbitrary translation surface X, C_X is rarely a Teichmüller curve. But for origamis, the situation is different:

Proposition 3.7. *For any origami $p : X \to E$ of genus g, C_X is a Teichmüller curve in M_g. More precisely, the Veech group Γ_X is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$.*

This was already observed by Thurston and Veech in the late 80's. Gutkin and Judge [**6**] showed that the converse also holds: any translation surface whose Veech group is commensurable to $\mathrm{SL}_2(\mathbb{Z})$, is an origami. A nice proof of the proposition and moreover a very useful characterization of the Veech group of an origami in terms of automorphisms of the free group F_2 of rank 2 can be found in G. Schmithüsen's thesis [**18**].

Proposition 3.8. *For any origami X, the Teichmüller curve C_X is defined over a number field.*

This follows from the previous proposition and Belyi's theorem: Since the only elliptic fixed points of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathbb{H}$ are the orbits of i and of $\rho = e^{2\pi i/3}$, the modular map $j : \mathbb{H} \to \mathbb{H}/\mathrm{SL}_2(\mathbb{Z}) = \mathbb{C}$ is ramified only above two points (namely 0 and 1728). Since Γ_X is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, it induces a finite covering $\mathbb{H}/\Gamma_X \to \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ which can be extended to a finite covering $q : \overline{\mathbb{H}/\Gamma_X} \to \mathbb{P}^1(\mathbb{C})$ of compact Riemann surfaces. q is ramified over at most three points (0, 1728 and ∞). By Belyi's theorem this implies that $\overline{\mathbb{H}/\Gamma_X}$ and hence also C_X is defined over a number field.

Möller [**16**], Thm. 5.1 has shown that every Teichmüller curve can be defined over a number field. The proof of this result requires quite different techniques.

4. p-adic uniformization and Mumford curves

A key point in the classical theory of Riemann surfaces is the uniformization theorem which implies that every compact Riemann surface of genus $g \geq 2$ can be conformally represented as the quotient of the upper half plane $\mathbb{H}$ by a discontinuous and free action of a Fuchsian group. More generally, any compact Riemann surface can be obtained in many ways as the quotient of an open subdomain of the Riemann sphere $\mathbb{C} \cup \{\infty\} = \hat{\mathbb{C}}$, which we identify with the projective line $\mathbb{P}^1(\mathbb{C})$, by a discontinuous group of Möbius transformations.

In this section we discuss to what extent an analog of this uniformization is possible over a p-adic field. We begin quite elementary by recalling the construction of the fields $\mathbb{Q}_p$ and $\mathbb{C}_p$:

4.1. p-adic valuation and p-adic fields. The *p-adic valuation* of an integer $a \neq 0$ is defined as $v_p(a) = n$ if a can be written as $a = p^n \cdot a'$, where p does not divide a'. In other words $v_p(a) = n$ if p^n divides a, but p^{n+1} does not. For a nonzero rational number $z = \frac{a}{b}$ we define $v_p(z) = v_p(a) - v_p(b)$, and the *p-adic absolute value* as $|z|_p = p^{-v_p(z)}$. Extending the map $|.|_p$ to all of $\mathbb{Q}$ by $|0|_p = 0$ we obtain a *norm* $|.|_p : \mathbb{Q} \to \mathbb{R}$. Besides the usual properties of a norm ($|x|_p = 0$ if and only if $x = 0$, $|xy|_p = |x|_p \cdot |y|_p$), it satisfies the "ultrametric" triangle inequality $|x+y|_p \leq \max(|x|_p, |y|_p)$. As do all norms, $|.|_p$ induces a metric on $\mathbb{Q}$ by $d_p(x,y) = |x-y|_p$. This metric is non-archimedean in the sense that

$$d_p(x,z) \leq \max(d_p(x,y), d_p(y,z)) \text{ for all } x, y, z \text{ in } \mathbb{Q}.$$

Recall that any non-archimedean metric implies the following "simplified" geometry:
(a) Any triangle is isosceles, i. e. has two sides of equal length; the third side then also has the same length or is shorter.
(b) Any two disks $B_{r_i}(x_i) = \{y | \, d_p(x_i, y) < r_i\}$, $i = 1, 2$, are either disjoint, or one of them is contained in the other.

One very important property of a non-archimedean norm $|\,.\,|$ on a field k is that the closed unit disk $\mathbb{D} = \overline{B}_1(0) = \{x \in k | \, |x| \leq 1\}$ is a ring $\mathcal{O}_k$, called the *valuation ring* of $|\,.\,|$. This holds because for x, y with $|x| \leq 1$ and $|y| \leq 1$ we also have $|x+y| \leq 1$ and $|-x| = |x| \leq 1$. Note that $\mathcal{O}_k$ is a local ring with the unique maximal ideal $\mathfrak{m} = \{x \in k | \, |x| < 1\}$. The quotient field $\kappa = \mathcal{O}_k/\mathfrak{m}$ is called the *residue field*; for $k = \mathbb{Q}$ and $|\,.\,|$ the p-adic absolute value, the residue field is the prime field $\mathbb{F}_p$.

Like for the euclidean norm, also for the p-adic norm $|\,.\,|_p$ the metric space $(\mathbb{Q}, |\,.\,|_p)$ is not complete. Denote by $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ with respect to $|\,.\,|_p$. It has the following properties:

Remark 4.1. (i) $\mathbb{Q}_p$ is a field which contains $\mathbb{Q}$ as a subfield.
(ii) The p-adic norm on $\mathbb{Q}$ extends to an absolute value $|\,.\,|_p$ on $\mathbb{Q}_p$ which takes on the same values as on $\mathbb{Q}$, namely the integer powers of p.
(iii) $\mathbb{Q}_p$ is complete w. r. t. $|\,.\,|_p$.

The second property holds because, due to the non-archimedean triangle inequality, every Cauchy sequence either converges to 0 or else has constant absolute value for sufficiently large n.

Since the p-adic norm on $\mathbb{Q}_p$ has no other values than on $\mathbb{Q}$, $\mathbb{Q}_p$ cannot be as close to an algebraically closed field as $\mathbb{R}$, the completion of $\mathbb{Q}$ w. r. t. the euclidean norm: e. g. not all the roots $p^{1/k}$ of p can be contained in one single finite field extension of $\mathbb{Q}_p$.

On the other hand, the residue field of $\mathbb{Q}_p$ is still $\mathbb{F}_p$. The valuation ring is $\mathbb{Z}_p$, which can also be obtained as the completion of $\mathbb{Z}$ w. r. t. $|\,.\,|_p$.

Let $\overline{\mathbb{Q}}_p$ be the algebraic closure of $\mathbb{Q}_p$. By standard arguments the absolute value on $\mathbb{Q}_p$ extends in a unique way to an absolute value on $\overline{\mathbb{Q}}_p$, which is still denoted $|\,.\,|_p$. The residue field of $\overline{\mathbb{Q}}_p$ is $\overline{\mathbb{F}}_p$, the algebraic closure of $\mathbb{F}_p$. Unfortunately $\overline{\mathbb{Q}}_p$ is no longer complete for the absolute value $|\,.\,|_p$, so we pass to the completion $\mathbb{C}_p$ of $\overline{\mathbb{Q}}_p$ w. r. t. $|\,.\,|_p$. It turns out that $\mathbb{C}_p$ is the analogue of the complex field $\mathbb{C}$ in the following sense:

PROPOSITION 4.2. (i) *$\mathbb{C}_p$ is a field extension of $\mathbb{Q}$, and the p-adic norm $|\,.\,|_p$ on $\mathbb{Q}$ extends in a unique way to $\mathbb{C}_p$.*
(ii) *$\mathbb{C}_p$ is complete w. r. t. $|\,.\,|_p$ and algebraically closed. The residue field of $\mathbb{C}_p$ is $\overline{\mathbb{F}}_p$.*

The drawback of the big extension from $\mathbb{Q}_p$ to $\mathbb{C}_p$ is that $\mathbb{C}_p$ is no longer locally compact, and the valuation ring is no longer a noetherian ring. Nevertheless, $\mathbb{C}_p$ allows for a theory of analytic functions in the sense of H. Cartan.

4.2. p-adic analytic functions. As in the complex world, convergent power series are the building blocks of analytic functions also over p-adic fields. A power series $\sum_{n\geq 0} a_n z^n$ with coefficients $a_n \in \mathbb{C}_p$ has the usual radius of convergence $r = (\limsup \sqrt[n]{|a_n|})^{-1}$ and thus converges (if at all) on some open disk $B_r(0) = \{z \in \mathbb{C}_p |\, |z| < r\}$. But since p-adic fields are totally disconnected and thus a disk $B_r(0)$ is a disjoint union of disks of radius $r' < r$, the classical definition of a holomorphic function as a function that can locally be expressed as a convergent power series, would not lead to a satisfactory theory: e. g. a locally constant function, which is 0 on some of the smaller disks and 1 on the others, would satisfy this definition.

To obtain a notion which shares the basic properties of complex holomorphic functions (like the identity theorem), one defines the holomorphic functions on a closed disk $\overline{B}_r(a)$ as the power series in $z - a$ that converge on that disk. Due to the non-archimedean valuation this is equivalent to the condition on the coefficients a_n of the series, that $|a_n|r^n$ converges to 0. Thus in particular, the holomorphic functions on $\overline{B}_1(0)$ form the "Tate algebra"

$$\mathbb{C}_p\!<\!z\!>= \{\sum_{n\geq 0} a_n z^n |\, a_n \to 0\}.$$

Similarly the holomorphic functions on $\overline{B}_r(a)$ are the elements of $\mathbb{C}_p\!<\!\frac{z-a}{c}\!>$, where $c \in \mathbb{C}_p$ is such that $|c| = r$.

Next we define the holomorphic functions on a domain $D \subset \mathbb{C}_p$ which is a "disk with holes", i. e. of the form

$$D = \overline{B}_r(a) - \bigcup_{i=1}^{m} B_{r_i}(a_i)$$

for some $a_i \in \overline{B}_r(a)$ and $r_i \le r$ as

$$A(D) = \mathbb{C}_p < \frac{z-a}{c}, \frac{c_1}{z-a_1}, \ldots, \frac{c_m}{z-a_m} >$$

with $|c_i| = r_i$.

Disks with holes are the simplest examples of (one-dimensional) *affinoid domains.* In general, affinoid domains correspond to "affinoid algebras", i. e. quotients of a Tate algebra $\mathbb{C}_p < z_1, \ldots, z_n >$ by certain ideals. The correspondence is such that the affinoid algebra is the ring of holomorphic functions on the affinoid domain. General p-adic analytic varieties are obtained by gluing affinoid domains in an "admissible" way. The admissible coverings in this sense include in particlar the finite ones. Good introductions to the one-dimensional theory can be found in [**5**] and [**4**], the general case is treated in [**1**].

Fortunately, in these notes we do not need more complicated affinoid domains than disks with holes. It should be noted however, that this is due to our limitation to Mumford curves: Exactly as in the complex case, every algebraic variety over $\mathbb{C}_p$ carries a natural structure as analytic variety. In dimension one, i. e. for projective algebraic curves over $\mathbb{C}_p$, it turns out, however, that almost no curve can "admissibly" be covered by disks, as is, in contrast, the case for all complex nonsingular projective curves ("admissibly" here would mean by finitely many of them). The basic insight, which goes back to Mumford [**17**], is that many, but by far not all curves over $\mathbb{C}_p$ can be covered by finitely many disks with holes. We call a projective nonsingular curve over $\mathbb{C}_p$ which, as an analytic variety, can be covered by finitely many disks with holes, a *Mumford curve.* In the following sections, we shall characterize these p-adic analogs of Riemann surfaces in different ways.

4.3. p-adic Schottky groups. The projective line $\mathbb{P}^1(\mathbb{C}_p) = \mathbb{C}_p \cup \{\infty\}$ is an analytic variety which can be covered by two disks. As usual, the group $\mathrm{PGL}_2(\mathbb{C}_p)$ of Möbius transformations acts on $\mathbb{P}^1(\mathbb{C}_p)$ by analytic automorphisms. If $\Gamma \subset \mathrm{PGL}_2(\mathbb{C}_p)$ is a finitely generated subgroup which acts discontinuously on some open subdomain $\Omega \subset \mathbb{P}^1(\mathbb{C}_p)$, then the quotient Ω/Γ inherits a structure of one-dimensional analytic variety exactly in the same way as it is a Riemann surface in the corresponding complex situation.

There is one kind of classical Kleinian groups that can be defined in the same way over p-adic fields, namely Schottky groups. Instead of recalling Schottky's classical construction we give the p-adic definition right away since it is literally the same:

DEFINITION 4.3. Let $g \ge 1$ and $D_i = \overline{B}_{r_i}(a_i)$, $i = 1, \ldots, 2g$ mutually disjoint closed disks in $\mathbb{C}_p$. Denote by C_i the "boundary" of D_i, i. e. $C_i = \overline{B}_{r_i}(a_i) - B_{r_i}(a_i) = \{z \in \mathbb{C}_p |\, |z - a_i| = r_i\}$ (note that, unlike the complex case, the "center" a_i of D_i is not unique and therefore C_i depends on the choice of a_i).
Choose $\gamma_i \in \mathrm{PGL}_2(\mathbb{C}_p)$, $i = 1, \ldots, g$, such that $\gamma_i(C_i) = C_{i+g}$ and $\gamma_i(D_i) = (\mathbb{P}^1(\mathbb{C}_p) - D_{i+g}) \cup C_{i+g} = \mathbb{P}^1(\mathbb{C}_p) - B_{r_{i+g}}(a_{i+g})$.
Then the subgroup Γ of $\mathrm{PGL}_2(\mathbb{C}_p)$ generated by $\gamma_1, \ldots, \gamma_g$ is called a *p-adic Schottky group.*

Clearly the definition can be extended to allow disks in $\mathbb{P}^1(\mathbb{C}_p)$ that contain ∞. A p-adic Schottky group has the same properties as a complex one:

THEOREM 4.4. *Let $\Gamma \subset \mathrm{PGL}_2(\mathbb{C}_p)$ be a Schottky group as in Definition 4.3. Then the following hold:*

(i) *Γ is a free group of rank g.*
(ii) *Let $F = \mathbb{P}^1(\mathbb{C}_p) - \bigcup_{i=1}^{2g} D_i$ and $\Omega = \bigcup_{\gamma\in\Gamma} \gamma(F)$. Then Γ acts properly discontinuously (and freely) on Ω, and F is a fundamental domain for this action.*
(iii) *The quotient space $X^{\mathrm{an}} = \Omega/\Gamma$ is the analytic space associated with a projective nonsingular algebraic curve X_Γ over $\mathbb{C}_p$ of genus g.*
(iv) *X_Γ is a Mumford curve.*

PROOF. (i) and (ii) are proved exactly in the same way as in the complex situation.
(iii) can be proved by constructing nonconstant Γ-invariant meromorphic functions on Ω, see [**5**]. The original result, in a more algebraic-geometric setting, is due to Mumford [**17**].
To prove (iv), note that X_Γ can be covered by one disk with $2g$ holes F' and $2g$ annuli: For $i = 1, \dots, 2g$ choose positive real numbers r_i^-, r_i' and r_i^+ satisfying $r_i^- < r_i < r_i' < r_i^+$ such that the closed disks $\overline{B}_{r_i^+}(a_i)$ are still mutually disjoint and the projection map from Ω to X^{an} is injective on each annulus $\Delta_i = \{z \in \mathbb{C}_p \,|\, r_i^- \leq |z - a_i| \leq r_i^+\}$. Finally let $F' = \mathbb{P}^1(\mathbb{C}_p) - \bigcup_{i=1}^{2g} B_{r_i'}(a_i)$. $\square$

The following result shows that p-adic Schottky groups play a very important role among the discontinuous subgroups of $\mathrm{PGL}_2(\mathbb{C}_p)$:

THEOREM 4.5. *Let $G \subset \mathrm{PGL}_2(\mathbb{C}_p)$ be a finitely generated subgroup which acts discontinuously on some nonempty open subset of $\mathbb{P}^1(\mathbb{C}_p)$. Then G contains a p-adic Schottky group Γ as a subgroup of finite index.*

This theorem can be obtained by looking at the action of G on the Bruhat-Tits tree and using the structure theorem for the fundamental group of a graph of groups from Bass-Serre theory. A different proof is contained in [**5**].
There is a second result that emphasizes the importance of Schottky groups; its proof relies on the results of the following section:

THEOREM 4.6. *Every Mumford curve can be obtained as the quotient of an open dense subset of $\mathbb{P}^1(\mathbb{C}_p)$ by a p-adic Schottky group.*

5. The Bruhat-Tits tree

There is a very useful tree associated with the field of p-adic numbers which at the same time helps to understand the action of $\mathrm{PGL}_2(\mathbb{Q}_p)$ and its subgroups on $\mathbb{P}^1(\mathbb{Q}_p)$ and also the "reduction mod p" of certain subsets of $\mathbb{P}^1(\mathbb{Q}_p)$ and analytic varieties. Algebraically it is a special case of the general concept of the Bruhat-Tits building for a reductive algebraic group over a local field. The group here is $\mathrm{PGL}_2(\mathbb{Q}_p)$, and the building is one-dimensional and turns out to be a tree.
The tree $T = T(\mathbb{Q}_p)$ is defined as follows: the vertex set $V(T)$ is the set of closed disks in $\mathbb{P}^1(\mathbb{Q}_p)$, i. e. $V(T) = \{\overline{B}_r(a) | \, a \in \mathbb{Q}_p), r \in p^{\mathbb{Z}}\}$. There is a (directed) edge between disks D and D' if $D \subset D'$ and D is maximal in D' w.r.t. inclusion, i. e. there is no disk D'' with $D \neq D'' \neq D'$ and $D \subset D'' \subset D'$.
Clearly this graph is connected since any two disks $D = \overline{B}_r(a)$ and $D' = \overline{B}_{r'}(a')$ are contained in a common larger disk, e. g. $\overline{B}_{|a-a'|}(a)$. Moreover, from every vertex $\overline{B}_r(a)$ there is exactly one directed edge going out, namely the one to $\overline{B}_{r\cdot p}(a)$. Thus

a path in T without backtracking can change its direction at most once. From this observation one deduces immediately that T is a tree.

REMARK 5.1. The ends of T correspond bijectively to the elements of $\mathbb{P}^1(\mathbb{Q}_p)$.

PROOF. An end is an equivalence class of rays in T, where "equivalence" in a tree means finite difference. From a given vertex v, there is a unique ray in T starting at v and going up all the time. All these rays (for different v) are equivalent and thus form a single end in T which we let correspond to $\infty \in \mathbb{P}^1(\mathbb{Q}_p)$. All other rays ultimately go down infinitely many times and thus correspond to a sequence of disks $D_1 \supset D_2 \supset \dots$ whose radii tend to 0. Therefore $\bigcap_{i=1}^{\infty} D_i$ is a single point in $\mathbb{Q}_p$, which corresponds to the end represented by the ray $D_1 \supset D_2 \supset \dots$. $\square$

Note that T is a regular tree of valency $p+1$; for the particular vertex $\mathbb{Z}_p = \overline{B}_1(0)$, the p smaller neighbors correspond to the residue classes $p\mathbb{Z}_p, 1+p\mathbb{Z}_p, \dots, (p-1)+p\mathbb{Z}_p$, while the unique larger neighbor is $\overline{B}_p(0) = \overline{B}_{|\frac{1}{p}|}(0)$. Similar descriptions can be given for all other vertices.

Any two elements a, b of $\mathbb{P}^1(\mathbb{Q}_p)$ define an *axis* $A(a,b)$ in T, namely the straight line in T whose two ends correspond to the points a and b. Any three points a, b, c define a unique vertex called the *median* $m(a,b,c)$; it is the intersection of $A(a,b)$, $A(a,c)$ and $A(b,c)$.

PROPOSITION 5.2. $\mathrm{PGL}_2(\mathbb{Q}_p)$ *acts in a way on* T *which is compatible with its action on* $\mathbb{P}^1(\mathbb{Q}_p)$ *(considered as the set of ends of* T*).*

PROOF. For a closed disk D in $\mathbb{Q}_p$ and an element $\gamma \in \mathrm{PGL}_2(\mathbb{Q}_p)$, $\gamma(D)$ is again a closed disk in $\mathbb{Q}_p$ if (and only if) $\gamma^{-1}(\infty) \notin D$. In this case γ maps the vertex D of T to the vertex $\gamma(D)$. Clearly inclusion is preserved by γ, hence edges are mapped to edges.

If $\gamma^{-1}(\infty) \in D$, then $\gamma(D) = \mathbb{P}^1(\mathbb{Q}_p) \setminus B_r(a)$ for some open disk $B_r(a)$; we then map the vertex D to the vertex $\overline{B}_r(a)$. It is an easy exercise to verify that this definition is compatible with inclusions and defines an action of $\mathrm{PGL}_2(\mathbb{Q}_p)$ on T.

It is clear from the definition that an element $a \in \mathbb{P}^1(\mathbb{Q}_p)$ corresponding to a ray R in T is mapped by $\gamma \in \mathrm{PGL}_2(\mathbb{Q}_p)$ to $\gamma(a)$, which corresponds to the ray $\gamma(R)$. $\square$

REMARK 5.3. Let $\gamma \in \mathrm{PGL}_2(\mathbb{Q}_p)$ be an element with two fixed points a, b in $\mathbb{P}^1(\mathbb{Q}_p)$.

a) If γ is hyperbolic, then γ acts on the axis $A(\gamma) = A(a,b)$ by nontrivial translation.

b) If γ is elliptic, it fixes the axis $A(a,b)$ pointwise.

PROOF. Conjugating γ with a suitable $\delta \in \mathrm{PGL}_2(\mathbb{Q}_p)$ we may assume $a = 0$ and $b = \infty$. Then $\gamma(z) = \lambda \cdot z$ for some $\lambda \in \mathbb{Q}_p \setminus \{0\}$. γ is hyperbolic iff $|\lambda| \neq 1$, in which case γ acts on $A(0,\infty)$ by translation by $\log_p|\lambda|$. If $|\lambda| = 1$, γ is elliptic, and $\gamma(B_r(0)) = B_r(0)$ for every r. $\square$

In the same way as for $\mathbb{Q}_p$, a tree $T(K)$ can be constructed for every finite field extension $K/\mathbb{Q}_p$. There is a unique way to extend the p-adic valuation to K. The value group of K is a finite extension, say of degree e, of the value group $p^{\mathbb{Z}}$ of $\mathbb{Q}_p$. Thus K is again a field with a discrete valuation, its field of integers $\mathcal{O}_K = \{z \in K \mid |z| \leq 1\}$ is a discrete valuation ring with the unique maximal ideal $\mathfrak{m}_K = \{z \in K \mid |z| < 1\}$. The residue field $\mathcal{O}_K/\mathfrak{m}_K$ is a finite field extension of $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$, say of degree f. The degree $n = [K : \mathbb{Q}_p]$ of K over $\mathbb{Q}_p$ satisfies the famous formula $n = e \cdot f$.

The tree $T(K)$ arises from $T = T(\mathbb{Q}_p)$ by subdividing each edge into e edges (of equal length), subsequently adding $p^f - p$ edges to each of the original vertices, and finally completing this to a regular tree of valency $p^f + 1$.

We also have to consider infinite field extensions of $\mathbb{Q}_p$ which arise as limits of finite ones, e. g. $\mathbb{C}_p$. In this case one of the numbers e and f, or both of them, tends to ∞. As long as e stays bounded, we still have a simplicial tree $T(K)$, but it is not locally finite if f is not finite. Note that for any finitely generated subfield K of $\mathbb{C}_p$, the degree e of the extension of value groups is finite. If e tends to ∞ the corresponding limit of trees is no longer a simplicial tree, but it exists as an $\mathbb{R}$-tree.

It should be mentioned that there is an alternative way of constructing the tree $T(K)$ (which is in fact the original one): the vertices are homothety classes of $\mathcal{O}_K$-lattices in K^2, and the edges correspond to classes with representatives that are neighbors for inclusion. This approach is used e. g. in Serre's classical book [**19**].

Proposition and Definition 5.4. *Let $\Gamma \subset \mathrm{PGL}_2(\mathbb{C}_p)$ be a finitely generated discontinuous subgroup and $K \subset \mathbb{C}_p$ the smallest subfield containing the fixed points of all elements of $\Gamma \setminus \{\mathrm{id}\}$. Then*
a) *K is a finitely generated field extension of $\mathbb{Q}_p$.*
b) *There is a minimal nonempty subtree $T(\Gamma)$ of $T(K)$ on which Γ acts; this action is without inversions of edges.*
c) *If Γ is finite, $T(\Gamma)$ is reduced to a single vertex.*
d) *If Γ is infinite, $T(\Gamma)$ is the union of the axes of the hyperbolic elements in Γ.*
e) *$T(\Gamma)/\Gamma$ is a finite graph.*

Proof. **a)** Let $\gamma_1, \ldots, \gamma_n \in \mathrm{GL}_2(\mathbb{C}_p)$ be representatives of generators of Γ and K_0 the subfield of $\mathbb{C}_p$ generated by the matrix entries of the γ_i. K_0 is finitely generated over $\mathbb{Q}_p$ and contains all matrix entries of all representatives of elements of Γ. The fixed points of an element of $\mathrm{PGL}_2(K_0)$ are solutions of quadratic equations over K_0. There is an extension of K_0 of degree 4 which contains the solutions to all quadratic equations over K_0: a quadratic extension of the value group and a quadratic extension of the residue field suffice.
b) follows from c) and d). These in turn follow from the following two lemmas valid in general for group actions on trees:

Lemma 5.5. *Let γ_1, γ_2 be elliptic elements without common fixed point in T. Then the product $\gamma_1\gamma_2$ is hyperbolic.*

Lemma 5.6. *Let $\gamma \in \mathrm{Aut}(T)$ and $v \in V(T)$ a vertex of T. Then the unique path in T connecting v and $\gamma(v)$ intersects the axis $A(\gamma)$.*

Lemma 5.5 shows that all elements of a finite subgroup of $\mathrm{PGL}_2(\mathbb{C}_p)$ must have a common fixed point; this proves **c)**.
Lemma 5.6 shows that any nonempty subtree of $T(K)$ on which Γ acts must contain the axes of all hyperbolic elements of Γ. Since their union is already a Γ-invariant subtree, **d)** follows.
e) A finite fundamental domain for the action of Γ on $T(\Gamma)$ is obtained as follows: let $\gamma_1, \ldots, \gamma_n$ be generators of Γ; for each i, let F_i be a segment of length l_i of the axis $A(\gamma_i)$, where l_i is the translation length of γ_i. Then the smallest subtree F of $T(\Gamma)$ containing $F_1, \ldots, F_n$ is finite and surjects onto $T(\Gamma)/\Gamma$. $\square$

The quotient graph $\bar{T}(\Gamma) = T(\Gamma)/\Gamma$ acquires the structure of a *graph of groups* as follows: For every vertex v of $\bar{T}(\Gamma)$ take a vertex $\tilde{v}$ in $T(\Gamma)$ representing v and let G_v be the stabilizer in Γ of the disk corresponding to $\tilde{v}$; since Γ is discontinuous, G_v is a finite group. For every edge e of $\bar{T}(\Gamma)$ let $\tilde{e}$ be an edge in T representing e and G_e the stabilizer of $\tilde{e}$ in Γ. Let v_1 and v_2 be the endpoints of e and $\tilde{v}_1'$, $\tilde{v}_2'$ the endpoints of $\tilde{e}$; then since Γ acts without inversions on $T(\Gamma)$, G_e is the intersection of the stabilizers of $\tilde{v}_1'$ and $\tilde{v}_2'$ in Γ. These stabilizers are conjugate to G_{v_1} resp. G_{v_2}, and thus G_e corresponds to well defined subgroups of G_{v_1} and G_{v_2}.
These data, the graph $\bar{T}(\Gamma)$ together with the vertex groups G_v, the edge groups G_e and the inclusions $G_e \hookrightarrow G_{v_i}$, constitute a graph of groups. The main theorem of Bass-Serre theory says that the group Γ can be recovered (as an abstract group) from these data as the "fundamental group" of the graph of groups, see [**19**], Thm. 13. In our situation, Γ can even be recovered as a subgroup of $\mathrm{PGL}_2(K)$ up to conjugation.

Note that if Γ is a Schottky group, all vertex and edge groups are trivial. For an arbitrary discontinuous group, the vertex groups are finite subgroups of $\mathrm{PGL}_2(K)$. Since K is a field of characteristic 0, $\mathrm{PGL}_2(K)$ has the same finite subgroups as $\mathrm{PGL}_2(\mathbb{C})$, namely the cyclic groups $\mathbb{Z}/n\mathbb{Z}$ and the dihedral groups D_n for all $n \geq 1$, and the symmetry groups of the platonic solids, A_4, S_4 and A_5. It turns out that there are not too many possibilities for an edge group to be a proper subgroup of the vertex groups of both its end points. A complete list of all finite graphs of groups that come from finitely generated discontinuous subgroups of $\mathrm{PGL}_2(K)$ can be found in [**8**].

6. Reduction

Another way of characterizing Mumford curves is via reduction. This can be done either in an algebraic-geometric way (which I shall mention briefly at the end of this section) or in an analytic way which I shall explain now.
The basic idea of analytic reduction is a geometric interpretation of reduction mod p of K-algebras, where K is a p-adic field. In the simplest situation, the geometric object to be reduced is the p-adic unit disk, which is the same as the ring $\mathbb{Z}_p$ of p-adic integers. Its algebraic reduction $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$ is geometrically interpreted as the points of the affine line over $\mathbb{F}_p$, thus in particular an affine variety.

Conceptually this interpretation runs as follows: recall from Section 4.2 that the holomorphic functions on the unit disk are the elements of the Tate algebra $\mathbb{Q}_p < z >$ of power series in the variable z with coefficients that tend to 0. For a power series $f = \sum a_n z^n$ we define its norm as $||f|| = \max\{|a_n| \mid n \geq 0\}$. Let $\mathbb{Q}_p^0 < z >$ be the subring of $\mathbb{Q}_p < z >$ of elements of norm ≤ 1, and $\mathbb{Q}_p^{00} < z >$ the ideal of elements of norm < 1. Then the quotient ring $\mathbb{Q}_p^0 < z > /\mathbb{Q}_p^{00} < z >$ is obviously isomorphic to the polynomial ring $\mathbb{F}_p[z]$, which should be seen as the ring of regular functions on the affine line over $\mathbb{F}_p$. If we replace, as we did in Section 4.2, $\mathbb{Q}_p$ by $\mathbb{C}_p$ we have to replace $\mathbb{F}_p$ by its algebraic closure $\bar{\mathbb{F}}_p$.

In Section 4.2 we have defined Mumford curves as those curves that can be covered by finitely many disks with holes. Therefore we need to understand the reduction of disks with holes.

Definition 6.1. Let $D \subset \mathbb{C}_p$ be a disk with holes and $A(D)$ the $\mathbb{C}_p$-algebra of holomorphic functions on D. Let $||.||$ be the norm on $A(D)$ defined as the maximal

absolute value of the coefficients (where each of the generators is given norm 1). Finally let $A^0(D)$ be the subring of $A(D)$ of elements of norm ≤ 1 and $A^{00}(D)$ the ideal in $A^0(D)$ of elements of norm < 1.
Then $\bar{A}(D) = A^0(D)/A^{00}(D)$ is a finitely generated $\bar{\mathbb{F}}_p$-algebra and hence the ring of regular functions of a unique (up to isomorphism) affine variety $\bar{D}$ over $\bar{\mathbb{F}}_p$. $\bar{D}$ is called the *reduction* of D.

EXAMPLE 6.2. **1)** Let $D = \overline{B}_1(0) \setminus B_1(0) = \{z \in \mathbb{C}_p \mid |z| = 1\}$. Then $A(D) = \mathbb{C}_p < z, \frac{1}{z} >$ and $\bar{A}(D) = \bar{\mathbb{F}}_p[z, \frac{1}{z}]$. Hence $\bar{D} = \bar{\mathbb{F}}_p \setminus \{0\}$.
2) $D = \overline{B}_1(0) \setminus B_{1/p}(0) = \{z \in \mathbb{C}_p \mid \frac{1}{p} \leq |z| \leq 1\}$. Here $A(D) = \mathbb{C}_p < z, \frac{p}{z} >= \mathbb{C}_p < Z_1, Z_2 > /(Z_1 Z_2 - p)$. It follows that $\bar{A}(D) = \bar{\mathbb{F}}_p[Z_1, Z_2]/(Z_1 Z_2)$ is the affine coordinate ring of the union of two lines that intersect in one point (the Z_1-axis and the Z_2-axis in the plane).

In the first example, we removed a disk which is a whole residue class (in the ring $\bar{\mathbb{Z}}_p$ of integers in $\mathbb{C}_p$). The "missing" point in the reduction $\bar{D}$ corresponds to this residue class.
In the second example, the residue class of 0 is not completely removed, but only part of it (namely a residue class mod p^2). This is reflected in the affine variety $\bar{D}$ by "blowing up" the point corresponding to the residue class of 0 to a whole projective line. The points of this projective line correspond to the residue classes mod p^2 which lie within the class of 0 mod p, and one additional point ("∞") corresponding to the elements in $\bar{\mathbb{Z}}_p$ that are not 0 mod p. Out of this projective line one point is missing, namely the one corresponding to the removed disk $B_{1/p}(0)$.

PROPOSITION 6.3. *Let $D \subset \mathbb{C}_p$ be a disk with holes. Then the reduction $\bar{D}$ is a "tree of affine lines" over $\bar{\mathbb{F}}_p$, i. e. an affine variety such that all irreducible components of $\bar{D}$ are affine lines (over $\bar{\mathbb{F}}_p$) and the dual graph is a tree (the vertices of the dual graph are the irreducible components of $\bar{D}$ and an edge is drawn for every point of intersection of two components).*

PROOF. The proof of this proposition is a nice exercise, mainly in classical algebraic geometry. □

The next step is to define the reduction of an analytic variety that can be covered by disks with holes. This will be done by gluing the reductions of the covering sets. It turns out that the reduction depends to some extent on the chosen covering. Since Definition 6.1 carries over literally to more general affinoid domains it is possible to define reductions for arbitrary p-adic analytic varieties, but we shall explain the construction in detail only for varieties that can be covered by disks with holes since this is sufficient in the context of Mumford curves.

REMARK 6.4. Let D_1, D_2 be two disks with holes in $\mathbb{C}_p$. If $D_1 \cap D_2 \neq \emptyset$, the intersection Y is another disk with holes. The inclusions $Y \subset D_i$ induce restriction homomorphisms $A^0(D_i) \to A^0(Y)$ and thus morphisms $\bar{Y} \to \bar{D}_i$ $(i = 1, 2)$ which are in fact inclusions.

We define the reduction of $D_1 \cup D_2$ with respect to the covering $\{D_1, D_2\}$ as the algebraic variety obtained by gluing $\bar{D}_1$ and $\bar{D}_2$ along $\bar{Y}$. Observe that $Z = D_1 \cup D_2$ is itself a disk with holes if $Y \neq \emptyset$ and that the reduction of $\bar{Z}$ of this disk with holes may be different from the reduction with respect to $\{D_1, D_2\}$, as the following example shows:

EXAMPLE 6.5. Let $D_1 = \overline{B}_1(0) \setminus B_{1/p}(0)$ and $D_2 = \overline{B}_{1/p}(0) \setminus B_{1/p^2}(0)$. We have seen above that each of $\bar{D}_1$ and $\bar{D}_2$ is the union of two lines intersecting in one point, and the same holds for the union $D_1 \cup D_2 = \overline{B}_1(0) \setminus B_{1/p^2}(0)$. On the other hand, the intersection is $Y = D_1 \cap D_2 = \{z \in \mathbb{C}_p | \, |z| = \frac{1}{p}\}$; by the first example, $\bar{Y}$ is a line with one point removed. In the glued variety, $\bar{Y}$ is an open part of a line which has one additional point in $\bar{D}_1$ and one in $\bar{D}_2$, hence is a projective line. The reduction thus has 3 irreducible components, two affine lines and one projective line; each of the affine lines intersects the projective line in one point (not both in the same), and there are no other intersections.

DEFINITION 6.6. Let X be a p-adic analytic variety und $\mathfrak{U} = (U_i)_{i \in I}$ an open covering of X by disks with holes. Then we define the *reduction* $\bar{X}$ *of* X *with respect to* $\mathfrak{U}$ to be the gluing of the affine varieties $\bar{U}_i$ along their intersections as above.

REMARK 6.7. If X is an open analytic subset of $\mathbb{C}_p$, $\bar{X}$ is a tree of (affine and) projective lines.

This follows from Proposition 6.3 and a generalization of Example 6.5.

EXAMPLE 6.8. Let $X = \mathbb{C}_p \setminus \{0\}$ and $q \in \mathbb{C}_p$ with $|q| > 1$. Then the annuli $U_i = \{z \in \mathbb{C}_p | \, |q|^i \leq |z| \leq |q|^{i+1}\}$ $(i \in \mathbb{Z})$ cover X. The reduction is an infinite "cyclic" (i. e. 2-regular) tree of projective lines.

This example generalizes to

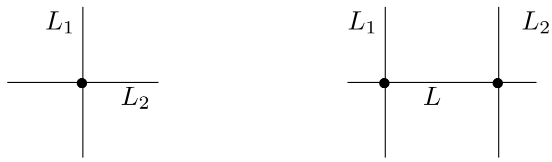
PROPOSITION 6.9. *Let* $\Gamma \subset \mathrm{PGL}_2(\mathbb{C}_p)$ *be a Schottky group,* F *a fundamental region as in Theorem 4.4 and* Ω *its region of discontinuity; the* $\gamma(F)$, $\gamma \in \Gamma$, *form an open covering of* Ω *by disks with holes. The reduction of* Ω *with respect to this covering is a tree of projective lines whose dual graph is isomorphic to the tree* $T(\Gamma)$ *defined in Proposition 5.4 (up to removing or inserting vertices of order 2).*

COROLLARY 6.10. *Let* X *be a Mumford curve of genus* g *over* $\mathbb{C}_p$. *Then* X *admits a finite open covering* $\mathfrak{U}$ *by disks with holes such that the reduction of* X *with respect to* $\mathfrak{U}$ *is a (singular) projective curve* $\bar{X}$ *over* $\bar{\mathbb{F}}_p$ *of arithmetic genus* g. *All irreducible components of* $\bar{X}$ *are rational curves, and all intersection points are ordinary double points.*

PROOF. Let $\Gamma \subset \mathrm{PGL}_2(\mathbb{C}_p)$ be a Schottky group with domain of discontinuity Ω such that $\Omega/\Gamma = X$. The reduction $\bar{\Omega}$ of Ω has a natural Γ-action, and $\bar{\Omega}/\Gamma$ is the reduction of X with respect to the covering used in the proof of Theorem 4.4 (iv).
Since Γ acts by translation on the irreducible components of $\bar{\Omega}$, identification of points on the same component can occur at most for intersection points with other components. Thus the irreducible components of $\bar{X}$ are projective lines on which possibly finitely many pairs of points are identified (at most g). This shows that all irreducible components of $\bar{X}$ have geometric genus 0. It follows that the arithmetic genus of $\bar{X}$ is just the genus (or first Betti number) of the intersection graph. By the proposition, this graph is (up to subdivision of edges) isomorphic to $\bar{T}(\Gamma)$. As the quotient graph of a tree by the free action of a free group of rank g, $\bar{T}(\Gamma)$ has genus g. □

Projective curves as in the corollary are called "totally degenerate". As mentioned before, reduction can be defined for all p-adic analytic varieties and thus in particular for all projective curves. Then Mumford curves can be characterized as those nonsingular projective curves over $\mathbb{C}_p$ whose reduction is totally degenerate.

The reduction of a variety is not unique. It is e. g. always possible to insert an additional annulus into the intersection of two disks with holes. In the reduction such an annulus corresponds to replacing the intersection point of two components L_1, L_2 by an additional component L which intersects L_1 and L_2 in the respective points where they originally intersected:



The inverse process is called "contracting the component L". It turns out that reduction is unique up to inserting or contracting components in this way. In particular, there is a unique "stable reduction" where no component can be contracted, i. e. all components are either singular or intersect other components in at least three points.

There is a purely algebraic way of defining the reduction of a projective curve X over $\mathbb{C}_p$: one has to find a "model" $\mathcal{X}$ of X over $\bar{\mathbb{Z}}_p$, the valuation ring of $\mathbb{C}_p$. $\mathcal{X}$ is a scheme over $\bar{\mathbb{Z}}_p$ such that X is obtained from $\mathcal{X}$ by base change: $X = \mathcal{X} \times_{\bar{\mathbb{Z}}_p} \mathbb{C}_p$. The reduction $\bar{X}$ is then defined as the base change with respect to the residue map $\bar{\mathbb{Z}}_p \to \bar{\mathbb{F}}_p$:

$$\bar{X} = \mathcal{X} \times_{\bar{\mathbb{Z}}_p} \bar{\mathbb{F}}_p.$$

Again, the model $\mathcal{X}$ and the reduction $\bar{X}$ are not unique, but there is a unique stable reduction. This follows from the famous work of Deligne-Mumford, see [**7**], Sect. 3C for an introduction. It is well known that the analytic and the algebraic stable reduction of a nonsingular curve agree, see e. g. [**2**].

If X is a plane curve given as the zero set of a homogeneous polynomial $F \in \mathbb{Q}_p[X, Y, Z]$, we may assume that the coefficients of F are in $\mathbb{Z}_p$, but not all in $p\mathbb{Z}_p$. Then the reduction $\bar{F} \in \mathbb{F}_p[X, Y, Z]$ of F mod p determines a projective curve over $\mathbb{F}_p$ which is a reduction of X (in most cases not the stable one).

7. p-adic origamis

In this section we combine the notion of an origami from Section 2 with the concept of Mumford curves. Recall that an origami over the field $\mathbb{C}_p$ is a finite morphism $p : X \to E$ of projective nonsingular curves over $\mathbb{C}_p$ such that E is of genus 1 and p is ramified over (at most) one point.

Definition 7.1. An origami $p : X \to E$ over $\mathbb{C}_p$ is called a *p-adic origami* if X and E are Mumford curves.

As usual, a covering $p : Y \to X$ is called *normal* (or *Galois*) if the group $\mathrm{Deck}(Y/X)$ of deck transformations acts transitively on the fibre $p^{-1}(x)$ for each $x \in X$. In this case, X is the quotient of Y by the subgroup $\mathrm{Deck}(Y/X)$ of the automorphism group $\mathrm{Aut}(Y)$ of Y. In his thesis [**12**], the main results of which are published in [**13**], K. Kremer gave a complete classification and description of all normal p-adic origamis. This section gives a brief account of his work.

7.1. Normal p-adic origamis. The starting point is the following observation:

PROPOSITION 7.2. *Let $p : X \to E$ be a normal p-adic origami, where X is a Mumford curve of genus $g > 1$. Then there is a discontinuous subgroup G of $\mathrm{PGL}_2(\mathbb{C}_p)$ and a normal subgroup Γ of G of finite index which is a Schottky group of rank g, such that $\Omega/G \cong E$ and $\Omega/\Gamma \cong X$, where $\Omega \subset \mathbb{P}^1(\mathbb{C}_p)$ is the region of discontinuity of G (and hence also of Γ).*

PROOF. Since X is a Mumford curve of genus g, by Thm. 4.6 there is a Schottky group $\Gamma \subset \mathrm{PGL}_2(\mathbb{C}_p)$ of rank g with $\Omega(\Gamma)/\Gamma \cong X$. By assumption, E is the quotient of X by a subgroup $\bar{G}$ of its automorphism group $\mathrm{Aut}(X)$. It is well known that $\mathrm{Aut}(X) \cong N(\Gamma)/\Gamma$ where $N(\Gamma)$ is the normalizer of Γ in $\mathrm{PGL}_2(\mathbb{C}_p)$ (cf. [**5**], VII.2). Then the inverse image G of $\bar{G}$ in $N(\Gamma)$ satisfies all the required properties. □

The group G in the proposition cannot be a Schottky group: since E is of genus 1, G would have to be a free group of rank 1, i. e. isomorphic to $\mathbb{Z}$, and thus could not have a free subgroup of rank > 1. This observation corresponds to the fact that p has to be ramified since any unramified covering of an elliptic curve is again an elliptic curve (by the Riemann-Hurwitz formula).
Ramification of the covering $\Omega/\Gamma \to \Omega/G$ occurs precisely in the fixed points of elements of $G \setminus \Gamma$ that lie in Ω. Since fixed points of hyperbolic elements are limit points, G must contain elliptic elements of finite order. The condition that p is ramified over only one point requires that all fixed points in Ω of elliptic elements of G lie in the same G-orbit. This is a serious restriction on G and Γ, as we shall see. But first we give an example that satisfies all conditions:

7.2. An Example. In this subsection we construct a normal p-adic origami whose Galois group is a dihedral group D_n.
Let $n \geq 3$ be an odd integer and $\zeta \in \mathbb{C}_p$ a primitive n-th root of unity. Consider the Möbius transformation $z \mapsto \zeta \cdot z$ as an element of $\mathrm{PGL}_2(\mathbb{C}_p)$, represented by the matrix $\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix}$. Furthermore let $\alpha \in \mathrm{PGL}_2(\mathbb{C}_p)$ be represented by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, i. e. $\alpha(z) = \frac{1}{z}$. Then δ and α generate a dihedral group of order $2n$, and α exchanges the two fixed points 0 and ∞ of δ.
Next let $\gamma \in \mathrm{PGL}_2(\mathbb{C}_p)$ be a hyperbolic element with fixed points 1 and -1, i. e. with the same fixed points as α. Let G be the subgroup of $\mathrm{PGL}_2(\mathbb{C}_p)$ generated by γ, δ and α. There is an obvious homomorphism $\tau : G \to D_n$, where D_n is the dihedral group $< t, s|\, t^n = s^2 = (ts)^2 = 1 >$: τ is given by $\tau(\delta) = t$, $\tau(\alpha) = s$ and $\tau(\gamma) = 1$. The kernel Γ of τ is the normal subgroup of G generated by γ. Since γ commutes with α, Γ is generated as a group by the n elements $\gamma_i = \delta^{i-1}\gamma\delta^{1-i}$, $i = 1, \dots, n$.
Let $d = \min\{|\zeta^i - 1|, |\zeta^i + 1| \mid i = 1, \dots, n-1\}$. As n is odd, $\zeta^i \neq -1$ for all i and hence $d > 0$; on the other hand $d \leq 1$ since $|\zeta| = 1$.
Being a hyperbolic element, γ is conjugate to $z \mapsto \lambda z$ for some $\lambda \in \mathbb{C}_p$ with $|\lambda| < 1$. Choose γ in such a way that $|\lambda| < d$, and assume for simplicity $p \neq 2$. The generators γ_i of Γ are all conjugate to $z \mapsto \lambda z$, therefore on the tree $T(\Gamma)$, γ_i acts on its axis $A(\gamma_i)$ by translation by $\log_p|\lambda^{-1}|$. Recall that the endpoints of

$A(\gamma_i)$ correspond to the fixed points of γ_i in $\mathbb{P}^1(\mathbb{C}_p)$, which are $\delta^{i-1}(1) = \zeta^{i-1}$ and $\delta^{i-1}(-1) = -\zeta^{i-1}$. By our choice of λ (and p), the intersection of $A(\gamma_i)$ and $A(\gamma_j)$ for $i \neq j$, which is of length at most $-\log_p(d)$, is shorter than the translation length of γ_i (and γ_j).

Let $T_0 \subset T(\Gamma)$ be the union of all intersections $A(\gamma_i) \cap A(\gamma_j)$ for $i \neq j$. Since $|\zeta^{i-1} - (-\zeta^{i-1})| = |2\zeta^{i-1}| = 1$ for all i (here we use $p \neq 2$), the vertex v_0 of $T(K)$ corresponding to the disk $\overline{B}_1(\zeta^{i-1}) = \overline{B}_1(0)$ lies on each $A(\gamma_i)$ and hence T_0 is connected.

Since $|\lambda| < d$ we can find a vertex v_i on $A(\gamma_i) \setminus T_0$ such that the segment from v_i to $\gamma(v_i)$ has nonempty intersection with T_0 (thus in particular contains v_0) and $\gamma(v_i) \notin T_0$. Let D_i and D_i' be the closed disks in $\mathbb{C}_p$ corresponding to the vertices v_i and $\gamma(v_i)$. By construction the $2n$ disks D_i, D_i', $i = 1, \ldots, n$, are mutually disjoint. They thus satisfy the properties of Definition 4.3, which shows that Γ is a Schottky group. As a consequence, G is also discontinuous.

Denote, as usual, by Ω the set of discontinuity of G and Γ. So far we have seen that $X = \Omega/\Gamma$ is a Mumford curve of genus g. Moreover we know (because G is discontinuous) that $E = \Omega/G$ is a Mumford curve, too, and that $p : X \to E$ is a normal covering of degree $2n$ with deck transformation group D_n.

Next we determine the ramification of the covering p. Clearly p is ramified precisely in the fixed points in X of the elements of the deck transformation group. These in turn are the images in X of the fixed points in Ω of the elements of the group generated by δ and α. Since the fixed points of α, and also those of the other involutions in this dihedral group, are limit points of Γ (they are fixed points of the hyperbolic elements $\delta^{i-1}\gamma\delta^{1-i}$), only the orbits of the two fixed points of δ give rise to ramification points of p. They are both of order $n = \mathrm{ord}(\delta)$. Thus we can read off the genus g_E of E from the Riemann-Hurwitz formula:

$$2n - 2 = 2n(2g_E - 2) + 2 \cdot (n - 1).$$

This shows $g_E = 1$. Moreover we know that the two fixed points of δ are in the same G-orbit since they are exchanged by α. This means that p maps them both to the same point in E. We have proved:

PROPOSITION 7.3. *With notations as above, $p : X = \Omega/\Gamma \to E = \Omega/G$ is a normal p-adic origami with Galois group D_n.*

We can also easily find the quotient graphs $T(\Gamma)/\Gamma$ and $T(\Gamma)/G = T(G)/G$: up to contraction of some edges in the case $|1 - \zeta| < 1$, $T(\Gamma)/\Gamma$ consists of a single vertex $\bar{v}_0$ (the image of v_0), and one loop for each of the n free generators of Γ. D_n acts in the following way on this graph: δ fixes the vertex and cyclically rotates the n loops, whereas α fixes one of the loops (the one corresponding to the axis of γ) and exchanges the remaining $n - 1$ loops in pairs (remember that n is odd!). Therefore the quotient graph $T(\Gamma)/G$, which is also the quotient graph of $T(\Gamma)/\Gamma$ by the action of D_n, consists of a single vertex v and a single edge e. As a graph of groups, the vertex group is D_n and the edge group is cyclic of order 2 (generated by α, say); it is embedded into D_n both ways (i.e. via e and via $\bar{e}$) as the same element, which reflects the fact that $\gamma\alpha\gamma^{-1} = \alpha$.

7.3. Kremer's results. By similar reasoning as in Section 7.2, Kremer found the following example: Let δ and α be elements of $\mathrm{PGL}_2(\mathbb{C}_p)$ of order 3 and 2, resp., that generate a tetrahedral group A_4, and let γ be a hyperbolic element commuting with δ. The group G generated by δ, α and γ can be mapped homomorphically onto A_4 in such a way that the kernel Γ is generated by γ and its conjugates. Γ is a Schottky group of rank 4, and $p : X = \Omega/\Gamma \to E = \Omega/G$ is a normal p-adic origami of degree 12. That p is ramified over only one point is due to the fact that A_4 acts transitively on the 6 fixed points of the elements of order 2 in A_4 (in the same way as A_4 acts transitively on the six edges of a tetrahedron). The quotient graph $T(G)/G$ in this case has, as above, one vertex and one edge, the vertex group being A_4 and the edge group being cyclic of order 3 generated by δ.

The main result of Kremer's paper is that essentially all normal p-adic origamis can be obtained from these two examples:

Theorem 7.4 (Thm. 5.1 in [**13**]). *Let $p : X = \Omega/\Gamma \to E = \Omega/G$ be a normal p-adic origami with genus$(X) > 1$. Then $T(G)/G$ can be contracted to a graph with one vertex and one edge, in other words, G is the fundamental group of a graph of groups with a graph of this type.*
If $p > 5$, the vertex group is isomorphic to either D_n for some $n \geq 3$ or to A_4. The edge group is cyclic of order 2 in the first case and of order 3 if the vertex group is A_4.
For $p = 2, 3$ and 5 there exist additional possibilities for the vertex group.
The group Γ can be any normal subgroup of G that has trivial intersection with the vertex group.

The proof of this theorem relies on precise knowledge of the possible discontinuous subgroups of $\mathrm{PGL}_2(\mathbb{C}_p)$ and on a careful analysis of the properties of the quotient graphs.

The theorem tells us, which finite groups occur as Galois groups of normal p-adic origamis: such a group has to be a homomorphic image of one of the groups G in the theorem, and it must contain a copy of the corresponding vertex group (i. e. of D_n or A_4 if $p > 5$). This still allows for a wide range of interesting groups, see [**12**], Ex. 4.17 for a few examples. On the other hand the characterization excludes many groups that occur as Galois groups of complex origamis since this is the case for every finite group that can be generated by two elements.

A very interseting question, which is also addressed in Kremer's paper, concerns the relation between complex and p-adic origamis. The question can be formulated in at least two ways:

1. Given a complex origami O, does there exist a p-adic origami on the Teichmüller curve determined by O?
2. Given a p-adic origami, is it possible to recover the combinatorial description by squares of the corresponding complex origami?

More precisely the first question should be formulated as follows: A complex origami $O = (p : X \to E)$ of genus g defines a Teichmüller curve $C(O)$ in the moduli space M_g of compact Riemann surfaces of genus g. Since M_g is an algebraic variety which can be defined over $\bar{\mathbb{Q}}$ (and even over $\mathbb{Q}$), and since $C(O)$ is defined over a number field, $C(O)$ also determines an algebraic curve in $M_g(\bar{\mathbb{Q}})$ and hence, by extension of coefficients, also in $M_g(\mathbb{C}_p)$. The question now is, whether this algebraic curve in $M_g(\mathbb{C}_p)$ contains a point that represents a p-adic origami.

There is no general method to decide this question for a given normal complex origami. Negative results can be obtained by showing that the Galois group of the origami is not a homomorphic image of one of the groups G in Thm. 7.4. Positive results are possible in the rare cases where explicit equations are known for the points on the Teichmüller curve $C(O)$: In [**13**], Example 7.1, Kremer uses a result of A. Kappes (see [**10**]) that describes the Teichmüller curve to a specific origami with 6 squares by hyperelliptic equations; for a hyperelliptic curve over a p-adic field, it depends only on the relative position of the branch points whether it is a Mumford curve or not. With this criterion, Kremer shows that there are p-adic origamis on this special Teichmüller curve.

For the second question, there are also only very partial results available: Given a p-adic origami in the form of Thm. 6.4, i. e. in terms of the uniformizing groups, we may assume that the corresponding Mumford curves are defined over a number field and thus determine a complex origami (because also p-adic origamis come in 1-parameter families which necessarily contain $\bar{\mathbb{Q}}$-points). One would like to describe this complex origami e. g. as a translation surface by gluing squares. Since we consider only normal origamis, there is sometimes an indirect way to fix the complex origami, namely if there is only one normal origami with the given Galois group. In [**12**] Kremer shows that this is the case for all but 30 out of the 2386 groups of order up to 250 which can be generated by two elements. But of course, for larger order there are more groups which occur as Galois groups of different origamis. Kremer also shows that each group in the infinite series $D_n \times \mathbb{Z}/m\mathbb{Z}$ and $A_4 \times \mathbb{Z}/m\mathbb{Z}$ uniquely determies a normal origami and thus provides a way to associate a complex origami to the corresponding p-adic origami.

References

1. S. Bosch, U. Güntzer and R. Remmert, Non-archimedean Analysis, *Grundlehren der mathematischen Wissenschaften* 261, Springer 1984.
2. S. Bosch and W. Lütkebohmert, Stable Reduction and Uniformization of Abelian Varieties I, *Math. Ann.* 270 (1985), 349–379.
3. A. Eskin, M. Kontsevich and A. Zorich, Lyapunov spectrum of square-tiled cyclic covers, *Preprint* 2010, arXiv:1007.5330.
4. J. Fresnel and M. van der Put, Rigid Analytic Geometry and its Applications, *Progress in Mathematics* 218, Birkhäuser 2003.
5. L. Gerritzen and M. van der Put, Schottky groups and Mumford curves. *Lecture Notes in Mathematics* 817, Springer 1980.
6. E. Gutkin and C. Judge, Affine mappings of translation surfaces, *Duke Math. J.* 103 (2000), 191–212.
7. J. Harris and I. Morrison, Moduli of Curves, *Graduate Texts in Mathematics* 187, Springer 1998.
8. F. Herrlich, p-adisch diskontinuierlich einbettbare Graphen von Gruppen, *Arch. Math.* 39 (3) (1982), 204–216.
9. F. Herrlich, Introduction to Origamis in Teichmüller Space, in: A. Papadopoulos (ed.) *Strasbourg Master class on Geometry*, Europ. Math. Soc. (2012), 233–253.
10. F. Herrlich, A. Kappes and G. Schmithüsen, A origami of genus 2 with a translation, *Preprint* 2008, arXiv:0805.1865.
11. P. Hubert and S. Lelièvre, Prime arithmetic Teichmüller disks in $\mathcal{H}_2$, *Israel J. Math.* 151 (2006), 281–321.
12. K. Kremer, Invariants of Complex and p-adic Origami-Curves, *PhD thesis, Universität Karlsruhe*, 2010.
13. K. Kremer, Normal origamis of Mumford curves, *manuscripta math.* 133 (2010), 83–103.
14. P. Lochak, On arithmetic curves in the moduli spaces of curves, *J. Inst. Math. Jussieu* 4(3) (2005), 443–508.
15. C. Matheus, M. Möller and J.-F. Yoccoz, A criterion for the simplicity of the Lyapunov spectrum of square-tiled surfaces, *Preprint* 2013, arXiv:1305.2033.

16. M. Möller, Variations of Hodge structure of Teichmueller curves, *Journal of the AMS* 19 (2006), 327–344.
17. D. Mumford, An analytic construction of degenerating curves over complete local fields *Compos. math.* 24 (1972), 129–172.
18. G. Schmithüsen, Veech Groups of Origamis. *PhD thesis, Universität Karlsruhe*, 2005.
19. J.-P. Serre, Trees, *Springer*, 1980.
20. W.A. Veech, Teichmüller curves in moduli space, Eisenstein series and an application to triangular billiards. *Inventiones Mathematicae*, 97(3) (1989), 553–583.

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