A comb of origami curves in M_3

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In this note we present some results of our joint work with Gabriela Schmithüsen on a very particular and, as we think, highly fascinating configuration of origami curves in the moduli space M_3. For most of the results, more details and proofs can be found in [2] and [4].

1 Origami curves

In this section we briefly recall the notions of origamis, Teichmüller embeddings, Veech groups, and Teichmüller curves. More details and background information can be found e. g. in G. Schmithüsen's article [9] in this volume.

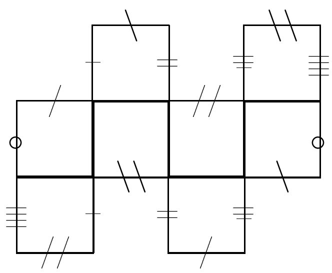
1.1 Origamis

An origami is a special kind of a translation surface. It can be obtained by the following combinatorial construction:

Take a finite number of (euclidean unit) squares and glue each left edge to a right edge and each top edge to a bottom edge, and vice versa.

If the resulting compact surface X is connected, we call it an *origami*. X is endowed with a translation structure which is obtained by using the squares as charts and translations for the gluing.

Our favourite example, which plays a crucial role in this note, is



If the edges are glued as indicated, i. e. edges with the same label are glued, we obtain a closed surface W with 4 marked points. The genus of W can be seen by counting squares (8), edges (16), and vertices (4): Euler's formula gives $8 - 16 + 4 = 2 - 2g$, thus $g = 3$.

Mapping each square of an origami onto the torus E defines a covering $X \to E$

of degree d (= the number of squares) which is unramified outside the vertices of the squares. Conversely, let $p : X \to E$ be a (ramified) covering and endow E with the (standard) translation structure η_I inherited from the one on $\mathbb{R}^2 = \mathbb{C}$ by the universal covering $E = \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$. Then η_I lifts to a translation structure $\mu_I = p^*(\eta_I)$ on $X^* = X - \{\text{ramification points of } p\}$. In the special situation that p is ramified over (at most) one point $\infty \in E$, μ_I decomposes X^* into squares. This motivates

Definition 1. *An* origami *is a finite covering $O = (p : X \to E)$ of the torus E which is ramified over at most one point $\infty \in E$.*

For an origami $O = (p : X \to E)$, we denote by X^* the inverse image of $E^* = E - \{\infty\}$ under p; thus $p : X^* \to E^*$ is an unramified covering.

1.2 Teichmüller embeddings

A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ defines a lattice $\Lambda_A = (a+ci)\mathbb{Z} \oplus (b+di)\mathbb{Z} \subset \mathbb{C}$. Identifying the torus E with $\mathbb{C}/\Lambda_A$ gives a translation structure η_A on E which can also be obtained from η_I by composing the chart maps with the affine map $A : \mathbb{R}^2 \to \mathbb{R}^2$.

An origami $O = (p : X \to E)$ thus defines a family $\mu_A = p^*(\eta_A)$ of translation structures on X^*, indexed by the matrices $A \in \mathrm{SL}_2(\mathbb{R})$.

Note that, since translations are holomorphic maps in $\mathbb{C}$, each of our translation structures μ_A defines a structure of Riemann surface on X^*, and also on X (since it can be extended in a unique way to the isolated points $p^{-1}(\infty)$. We take (X, μ_I) as reference Riemann surface of the Teichmüller space T_g of marked Riemann surfaces of genus $g = \mathrm{genus}(X)$; for each $A \in \mathrm{SL}_2(\mathbb{R})$, we consider the identity map $\mathrm{id} : X \to X$ as a marking of the Riemann surface (X, μ_A). This gives us a map

$$\tilde{f}_O : \mathrm{SL}_2(\mathbb{R}) \to T_g.$$

If $A \in \mathrm{SO}_2(\mathbb{R})$, the associated affine map is conformal, hence holomorphic. This shows that $\tilde{f}_O$ induces a map

$$f_O : \mathrm{SO}_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R}) \to T_g.$$

We identify $\mathrm{SO}_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R})$ with the complex upper half plane $\mathbb{H}$: $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by Möbius transformations, and $\mathrm{SO}_2(\mathbb{R})$ is the stabilizer of i. We then map the coset $\mathrm{SO}_2(\mathbb{R}) \cdot A$ to $-\overline{A^{-1}(i)}$.

There is the following basic

Fact 2. *An origami $O = (p : X \to E)$ induces an injective map*

$$f_O : \mathbb{H} \to T_g, \qquad g \text{ the genus of } X,$$

which is holomorphic and isometric (with respect to the hyperbolic metric on $\mathbb{H}$ and the Teichmüller metric on T_g).

Definition 3. *A holomorphic isometric embedding $\iota : \mathbb{H} \to T_g$ is called a* Teichmüller embedding. *In this case, $\Delta_\iota := \iota(\mathbb{H})$ is called a* Teichmüller disk.
If O is an origami, we write Δ_O instead of Δ_{f_O}.

1.3 Veech groups

For an origami $O = (p : X \to E)$ denote by $\mathrm{Aff}^+(O)$ the group of orientation preserving diffeomorphisms of X^* which are affine with respect to the translation structure μ_I. For an element $f \in \mathrm{Aff}^+(O)$, the linear part is a matrix $A \in \mathrm{SL}_2(\mathbb{R})$ which is independent of the charts and thus defines a homomorphism

$$D : \mathrm{Aff}^+(O) \to \mathrm{SL}_2(\mathbb{R}).$$

The kernel of D is the (finite!) group of translations of (X^*, μ_I).

Definition 4. *The image $\Gamma(O) = \mathrm{Aff}^+(O)/\ker(D) \subset \mathrm{SL}_2(\mathbb{R})$ is called the* Veech group *of O.*

For a different description of the Veech group recall that the mapping class group Mod_g acts on the Teichmüller space T_g by holomorphic isometries. The action is properly discontinuous, and the orbit space

$$M_g = T_g/\mathrm{Mod}_g$$

is the moduli space of Riemann surfaces of genus g, a quasi-projective algebraic variety of (complex) dimension $3g - 3$.
With an origami O we associate the subgroup

$$S(O) = \{\varphi \in \mathrm{Mod}_g : \varphi(\Delta_O) = \Delta_O\}$$

of elements of Mod_g that stabilize the Teichmüller disk Δ_O. Working carefully through the definitions and using results proved in [1] (see also [3, Section 2.4.3]), one finds the following important

Fact 5. *For an origami O we have*

$$\begin{array}{rcll} \mathrm{Aff}^+(O) & \cong & S(O) & \text{and} \\ \Gamma(O) & \cong & S(O)/S_0(O), & \\ \text{where}\quad S_0(O) & = & \{\varphi \in \mathrm{Mod}_g : \varphi|\Delta_O = \mathrm{id}_{\Delta_O}\}. & \end{array}$$

In [8] G. Schmithüsen found a very nice and useful alternate characterization of the Veech group of an origami. It is based on the following observation: If

$O = (p : X \to E)$ is an origami, the unramified covering $p : X^* \to E^*$ induces an embedding

$$U := \pi_1(X^*) \subset \pi_1(E^*) = F_2$$

of the fundamental groups, where F_2 is the free group on two generators. Then Schmithüsen proved the following

Fact 6.

$$\Gamma(O) = \mathrm{proj}(\mathrm{Stab}(U)),$$

where $\mathrm{Stab}(U) = \{\gamma \in \mathrm{Aut}^+(F_2) : \gamma(U) = U\}$ *is the stabilizer of* U*, and*

$$\mathrm{proj} : \mathrm{Aut}^+(F_2) \to \mathrm{Out}^+(F_2) = \mathrm{SL}_2(\mathbb{Z})$$

is the natural projection.

An immediate corollary is that $\Gamma(O)$ is always a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index (a fact that was known previously by different arguments).

1.4 Origami curves

For an origami O, denote by $C(O)$ the image of Δ_O in the moduli space $M_g = T_g/\mathrm{Mod}_g$. By Fact 2, the projection $\Delta_O \to C(O)$ factors through the Riemann surface $\tilde{C}(O) := \mathbb{H}/\Gamma(O)$ (to be precise, we have to define $\tilde{C}(O)$ as the mirror image of $\mathbb{H}/\Gamma(O)$, see [6]). Since $\Gamma(O)$ has finite index in $\mathrm{SL}_2(\mathbb{Z})$ and thus is a lattice in $\mathrm{SL}_2(\mathbb{R})$, $\tilde{C}(O)$ is of finite type. On the other hand, it is not compact, since $\Gamma(O)$ necessarily contains parabolic elements.
Another important result is (cf. [6, Cor. 3.3])

Fact 7. *For an origami* O*, the map* $\tilde{C}(O) \to C(O)$ *is birational.*

It follows that $C(O)$ is an algebraic curve (embedded into M_g); we call it the *origami curve* associated with O.

We have seen that $C(O)$ is never projective (thus always has cusps), and that $\tilde{C}(O)$ is its normalization. There are a few more general results on origami curves (but not too many):

- $C(O)$ is defined over a number field.

In fact, the inclusion $\Gamma(O) \subset \mathrm{SL}_2(\mathbb{Z})$ induces a finite covering $\tilde{C}(O) \to \mathbb{A}^1 = \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ which is ramified at most over two points (namely 0 and 1728). The result therefore follows from Belyi's theorem.

Remark: For any Teichmüller disk $\Delta_\iota \subset T_g$, one can consider its image in M_g. If it is an algebraic curve (which is a rather rare case) this curve is called a *Teichmüller curve.* Möller proved in [7] that all Teichmüller curves are defined over number fields.

- There are origami curves of arbitrarily large genus.

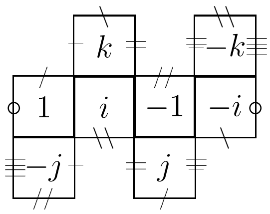
Examples were given by G. Schmithüsen in [8] where she showed for the infinite sequence O_k (k odd) of "cross shaped" origamis, that their Veech group is conjugated to the congruence group $\Gamma_1(2k)$.

- Even in M_2 there are origami curves of arbitrarily large genus.

This was shown by Hubert and Lelièvre [5] using L-shaped origamis.

2 The quaternion origami

In this section we present some results on the origami W which was investigated in detail in [2]. As an origami, W was already shown in Section 1.1. Here we first observe that W is closely related to the classical quaternion group Q with the eight elements $\pm 1, \pm i, \pm j, \pm k$ which are subject to the relations $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$. In fact, if we label the squares of W with the elements of Q as in the following picture,



we recognize the Cayley graph of Q: the right neighbour of the square labeled g is labeled $g \cdot i$, and the top neighbour is $g \cdot j$; for a description of origamis by Cayley graphs in the general situation see the last paragraph of Section 1 in [9].

Q acts on W by multiplying the labels from the left. This gives not only automorphisms of the Cayley graph, but also translations of the origami. In particular, the origami map $p : W \to E$ is a normal covering with Galois group Q.

The origami W has a lot of remarkable properties, some of which are listed in the following (see [2] for complete proofs):

- The Veech group of W is equal to $\mathrm{SL}_2(\mathbb{Z})$.

Using Fact 6 it suffices to show that the subgroup U of F_2 associated to W is a characteristic subgroup. Since $p : W^* \to E^*$ is normal with deck transformation group Q, U is the kernel of the homomorphism $F_2 \to Q$ that maps the generators x and y to i and j, respectively. One then shows that this kernel is stable under all automorphisms of F_2.

- The quotient of W by the subgroup $\{\pm 1\}$ of Q is E, and

$$p = [2] \circ q,$$

where $q : W \to W/\{\pm 1\}$ is the quotient map, and [2] is the multiplication by 2 on the elliptic curve E. Here we choose the origin ∞ for the group law on E to be one of the vertices of the squares; then the other three vertices are the points of order 2.

- The origami curve $C(W) \subset M_3$ is isomorphic to the affine line $\mathbb{A}^1$.

It follows from Fact 7 that there is a birational map

$$f : \mathbb{A}^1 = \mathbb{H}/\mathrm{SL}_2(\mathbb{Z}) = \mathbb{H}/\Gamma(W) \to C(W).$$

That in this special case, f can be shown to be an isomorphism, is essentially due to the following explicit description of $C(W)$:

- $C(W)$ is the image in M_3 of the family

$$W_\lambda : \quad y^4 = x(x-1)(x-\lambda), \qquad \lambda \in \mathbb{C} - \{0,1\}.$$

The proof of this result is based on the existence of an automorphism c of W of order 4 with 4 fixed points: Then $W/ < c >$ has genus 0, and an analysis of the monodromy leads to the equation.

The automorphism c fixes all vertices of W, and acts by rotation by π around the vertices (note that the total angle at a vertex is 4π since all 8 squares are glued to each vertex). c^2 is the translation $-1 \in Q$; c commutes with i and j, and the six automorphisms $\pm\sigma$, $\pm\rho$, and $\pm\tau$ all have order 2, where $\sigma = -ck$, $\rho = -ci$ and $\tau = -cj$.

Thus the automorphism group of W consists of the 8 translations in Q and the 8 elements $\pm c$, $\pm\sigma$, $\pm\rho$, and $\pm\tau$, all of which have derivative $-I$. We have found:

- There is an exact sequence of groups

$$1 \to Q \to \mathrm{Aut}(W) \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

The automorphism σ fixes the centers of the squares $1, -1, k$ and $-k$; similarly each of the other five involutions in $\mathrm{Aut}(W) - Q$ has 4 fixed points on W. This implies that $W/ < \sigma >$ has genus 1. In fact, we have the much stronger result

- For each $\lambda \in \mathbb{C} - \{0,1\}$, $W_\lambda / < \sigma > \cong E_{-1}$

where E_{-1} is the elliptic curve with equation $y^2 = x(x-1)(x+1) = x^3 - x$.

Since c commutes with σ, it descends to an automorphism $\bar{c}$ of $W_\lambda / < \sigma >$. But E_{-1} is the only elliptic curve with an automorphism of order 4 which has a fixed point.

The main result of [2] is based on the following

Observation. The quotient map $\kappa_\lambda : W_\lambda \to W_\lambda/ < \sigma > = E_{-1}$ is ramified over 4 points P, Q, R, S on E_{-1} that form an orbit under the automorphism $\bar{c}$. If P, Q, R, S all are n-torsion points for some $n \geq 2$, then $[n] \circ \kappa_\lambda : W_\lambda \to E_{-1}$ is ramified only over ∞ and hence an origami.
Since $\bar{c}$ preserves the kernel $E_{-1}[n]$ of multiplication by n, it even suffices for $[n] \circ \kappa_\lambda$ to be an origami that P is an n-torsion point.

Theorem 8. *For every $n \geq 3$ and every n-torsion point $P \in E_{-1}[n]$ on E_{-1}, there is $\lambda \in \mathbb{C} - \{0, 1\}$ such that P is a branch point of κ_λ.*
Then $D_P := [n] \circ \kappa_\lambda$ is an origami.

Corollary 9. *$C(W)$ intersects infinitely many other origami curves.*

We call this configuration in the moduli space – one algebraic curve intersected transversally by countably many others – a *comb*, although it will turn out in the next section that in our case, the picture of a comb should not be taken too literally.

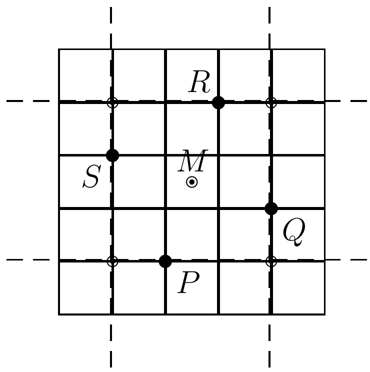
3 The comb embedded into a Hurwitz space

3.1 The origamis D_P

The origamis D_P that define the teeth of our comb are obtained from 2-sheeted coverings of the elliptic curve E_{-1} that are ramified over 4 points P, Q, R, S, all of which are n-torsion points for some $n \geq 3$. Moreover, P, Q, R and S are an orbit under the automorphism $\bar{c}$, thus $Q = \bar{c}(P)$, $R = \bar{c}^2(P)$ and $S = \bar{c}^3(P) = \bar{c}^2(Q)$.

As a Riemann surface, E_{-1} is isomorphic to $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, i. e. the ellpitic curve with the translation structure that we called η_I in Section 1.1 Since $\bar{c}$ has order 4, it lifts to rotation by $\frac{\pi}{2}$ around the origin on the universal covering $\mathbb{C}$. $\bar{c}$ has 2 fixed poitns on E_{-1}, namely ∞ (the image of the origin in $\mathbb{C}$) and M, the image of the midpoint $\frac{1}{2} + \frac{1}{2}i$ of the unit square. Note that M is a point of order 2 on E_{-1}.

A typical picture (with $n = 5$) of the configuration of the ramification points P, Q, R, S defining such an origami D_P is therefore



Note that $\bar{c}^2$ is rotation by π around ∞, which is the same as multiplication by -1. In particular $R = -P$ and $S = -Q$.

If we go to a different point on the origami curve $C(D_P)$, we replace the squares by parallelograms defined by a matrix $A \in \mathrm{SL}_2(\mathbb{R})$; on the torus E we then have the translation structure η_A. Thereby the automorphism $\bar{c}$ is transformed to a diffeomorphism $\bar{c}_A$ on $E_A = (E, \eta_A)$ which still has order 4, but in general is no longer conformal: $\bar{c}$ has derivative $D(\bar{c}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SO}_2(\mathbb{R})$, but $D(\bar{c}_A) = A \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A^{-1}$ which is in general not orthogonal.

On the other hand, $\bar{c}^2$ has derivative $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, which is central in $\mathrm{SL}_2(\mathbb{R})$. Thus $\bar{c}_A^2$ is multiplication by -1 on the elliptic curve E_A for every $A \in \mathrm{SL}_2(\mathbb{R})$.

The point on $C(D_P)$ corresponding to A therefore determines a 2-fold covering $X_A \to E_A$ which is ramified over 4 points P_A, Q_A, R_A, S_A satisfying $R_A = -P_A$ and $S_A = -Q_A$. In contrast to this, the condition "$Q_A = \bar{c}(P_A)$" cannot be expressed as an algebraic relation between P_A and Q_A.

3.2 The Hurwitz space H

In this section we study all coverings of the type just described in the previous section. More precisely we consider pairs (X, p) where

> X is a compact Riemann surface of genus 3
> $p : X \to E$ is a morphism of degree 2
> E is an elliptic curve
> p is ramified over 4 points $P, Q, -P, -Q$.

Note that the phrase "E is an elliptic curve" means that a point ∞ on E has been chosen as origin for the group law, so that multiplication by -1 is well defined.

Two such pairs (X, p) and (X', p') are considered equivalent, if there are isomorphisms $\varphi : X \to X'$ and $\bar{\varphi} : E \to E'$ satisfying $\bar{\varphi} \circ p = p' \circ \varphi$. In this case φ necessarily maps the ramification points of p to those of p', and $\bar{\varphi}$ maps $\{P, Q, -P, -Q\}$ onto $\{P', Q', -P', -Q'\}$.

Denote by $\tilde{H}$ the set of equivalence classes of pairs (X, p) as above. $\tilde{H}$ carries a natural structure of an algebraic variety. This is true much more generally for the set of equivalence classes of coverings $p : X \to Y$ of curves, where the genus of X and Y and the degree of p are fixed, and possibly also some additional data like ramification orders or the monodromy of p. These algebraic varieties are known as *Hurwitz spaces*. A general feature of these Hurwitz spaces is that the forgetful map $[p : X \to Y] \mapsto [X]$ gives a finite morphism to the moduli space M_g, where g is the genus of X.

In our particular situation, the algebraic structure on $\tilde{H}$ can be described very explicitly, as we shall see in the next section.

Note that $\tilde{H}$ is three dimensional, as can be seen by a naïve dimension count: one parameter to determine the elliptic curve, one for P, and one for Q.

We begin our investigation of $\tilde{H}$ with the following basic observation (see [4, Prop. 8]

Proposition 10. *Let (X, p) be a point in $\tilde{H}$. Then* $\mathrm{Aut}(X)$ *contains a subgroup isomorphic to the Klein four group V_4.*

Since p is of degree 2, it is the quotient for an involution σ on X. The key point in the proof of the proposition is to show that the automorphism $[-1]$ on E lifts to a second involution τ on X.

By the definition of $\tilde{H}$, σ has 4 fixed points on X (namely the inverse images of $P, Q, -P$ and $-Q$). The only possible fixed points of τ (and also of $\sigma\tau$) are the inverse images of the 4 fixed points of $[-1]$ on E (i. e. $p^{-1}(E[2])$). Each of these 8 points is fixed by either τ or $\sigma\tau$ (but not by both!). If one of τ or $\sigma\tau$ fixes all 8 points, the quotient by this automorphism has genus 0, which implies that X is hyperelliptic. If τ and $\sigma\tau$ fix 4 points each, all quotients have genus 1, and X is not hyperelliptic.

If the covering $p : X \to E$ is induced by an origami D_P as in Section 3.1, τ or $\sigma\tau$ coincides with c^2 (since they all descend to $[-1]$ on E). Since c^2 has 4 fixed points (the inverse images of ∞ and M), the above reasoning shows that X is not hyperelliptic. We therefore restrict our attention to the subset

$$H := \{(X, p) \in \tilde{H} : X \text{ is not hyperelliptic}\}$$

of $\tilde{H}$ (in fact, H is a connected component of $\tilde{H}$).

So far we have seen in this section that the comb $\mathcal{C}$, which is the union of the origami curves $C(W)$ and $C(D_P)$ for all $P \in E_{-1}[n]$ for some $n \geq 3$, is contained in $\mu(H)$, where we denote by

$$\mu : H \to M_3$$

the forgetful map $[(X, p)] \mapsto [X]$. The surprising result now is, see [4, Thm. 1]:

Theorem 11. *$\mathcal{C}$ is dense in $\mu(H)$.*

Since μ is a finite morphism of algebraic varieties, the inverse image $C_P := \mu^{-1}(C(D_P)) \subset H$ is also an algebraic curve (for each $P \in E_{-1}[n]$, $n \geq 3$). An equivalent formulation of Theorem 11 is therefore

Theorem 11′. *The union of the C_P, P an n-torsion point on E_{-1} for some $n \geq 3$, is dense in H.*

The idea of the proof is to approximate, for given $(X, p) \in H$, the points P and Q on the elliptic curve $E = p(X)$ by torsion points P_n, Q_n. This gives points in H that approximate (X, p) and lie on some origami curve. The technical heart of the proof is then to find $A \in \mathrm{SL}_2(\mathbb{R})$ such that $E = E_A$ and $Q_n = \bar{c}_A(P_n)$ (i. e. $P_n, Q_n, -P_n, -Q_n$ is an orbit under the affine deformation $\bar{c}_A$ of $\bar{c}$).

3.3 Affine coordinates for H

In this final section we sketch the explicit description of the Hurwitz space H introduced in the last section.

For this purpose we consider the family

$$C_{abc}: \quad x^4 + y^4 + z^4 + 2ax^2y^2 + 2bx^2z^2 + 2cy^2z^2$$

of plane projective complex curves. A straightforward calculation shows that C_{abc} is singular if one of a, b or c is $+1$ or -1, or if $a^2 + b^2 + c^2 - 2abc = 1$, and nonsingular otherwise. Therefore we introduce the open subset

$$U := \{(a, b, c) \in \mathbb{C}^3 : (a^2 - 1)(b^2 - 1)(c^2 - 1)(a^2 + b^2 + c^2 - 2abc - 1) \neq 0\}$$

of $\mathbb{C}^3$. For $(a, b, c) \in U$, the nonsingular curve C_{abc} is not hyperelliptic (no hyperelliptic curve can be smoothly embedded into the projective plane).

C_{abc} admits the automorphisms

$$\begin{array}{rcll} \alpha : (x : y : z) & \mapsto & (-x : y : z) & \quad \text{and} \\ \beta : (x : y : z) & \mapsto & (x : -y : z), & \end{array}$$

which generate a group isomorphic to V_4. α has 4 fixed points (namely the points $(0 : y : 1)$ with $y^4 + 2c^2 + 1 = 0$); they are symmetric with respect to β. This shows that the quotient map

$$p_\alpha : C_{abc} \to C_{abc}/ {<}\alpha{>}$$

defines a point in H. Conversely, each $(X, p) \in H$ can be represented by some C_{abc} (see [4, Prop. 11]), so we have

Proposition 12. *The map*

$$h : U \to H, (a, b, c) \mapsto (C_{abc}, p_\alpha)$$

is a finite surjective morphism of algebraic varieties.

It follows that the natural map

$$m : U \to M_3, (a, b, c) \mapsto [C_{abc}]$$

factors through H.

On the other hand, m also factors through the quotient U/L by the group L of linear automorphisms of U. Clearly L contains all permutations of a, b and c, and also the map v with $v(a,b,c) = (-a,-b,c)$. It is easy to see that L is generated by these automorphisms, and that L is isomorphic to S_4.

Thus we have the following commutative diagram of finite morphisms of algebraic varieties:

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "\tilde{q}"'] \arrow[dr, "m"] & H \arrow[d, "\mu"] \\ U/L \arrow[r, "q"'] & M_3 \end{tikzcd}$$

Theorem 13. a) *U/L is nonsingular.*
b) *q is birational.*

Proof. b) is shown in [4, Prop. 16]. One first observes that μ has degree 3 (because for a general point (X,p) in H, $\mathrm{Aut}(X) = V_4$, thus X has exactly 3 involutions). By direct computation one then shows that the degree of h is 8. Since the degree of $\tilde{q}$ is $|L| = 24$, it follows from the commutativity of the diagram that the degree of q is 1.
a) We prove the stronger result that $\mathbb{C}^3/L$ is nonsingular, more precisely:

$$\mathbb{C}^3/L \cong \mathbb{C}^3$$

Thus we have to show that the subalgebra $A := \mathbb{C}[a,b,c]^L$ of L-invariant polynomials in a, b and c is isomorphic to the polynomial ring in three variables. For this it suffices to show that A can be generated by 3 elements, since the transcendence degree of A is the same as that of $\mathbb{C}[a,b,c]$.
Obviously we have x, y and z in A, where

$$\begin{aligned} x &= a^2+b^2+c^2 \\ y &= abc \\ z &= a^2b^2+a^2c^2+b^2c^2 \end{aligned}$$

Now let f be any homogeneous L-invariant polynomial.
We claim that for any monomial $a^ib^jc^k$, that occurs in f with nonzero coefficient, the three exponents i, j and k all have the same parity.
If not, we may assume that $j \equiv k \bmod 2$, but $i \not\equiv j \bmod 2$. But then $v(a^ib^jc^k) = -a^ib^jc^k$, contradicting the invariance of f.
Now assume that f contains a monomial $a^ib^jc^k$ with a, b and c odd. Then

$$\tilde{f} := \sum_{g\in L} g(a^ib^jc^k)$$

is L-invariant and divisible by y:

$$\tilde{f} = abc \sum_{g \in L} g(a^{i-1} b^{j-1} c^{k-1})$$

where now all exponents of all monomials in the sum are even.

For suitable $\lambda \in \mathbb{C}^{\times}$, $f - \lambda \tilde{f}$ does not contain the monomial $a^i b^j c^k$ (and is still invariant).

We are therefore reduced to the case that for all monomials occuring in f, all three exponents are even. But then f is a symmetric polynomial in a^2, b^2 and c^2 and thus can be expressed by the elementary symmetric polynomials in a^2, b^2 and c^2, which are x, z, and y^2. $\square$

References

[1] C. Earle and F. Gardiner: *Teichmüller disks and Veech's F-structures.* American Mathematical Society. Contemporary Mathematics 201 (1997), 165–189.

[2] F. Herrlich and G. Schmithüsen: *An extraordinary origami curve.* To appear in Math. Nachrichten.

[3] F. Herrlich and G. Schmithüsen: *On the boundary of Teichmüller disks in Teichmüller and in Schottky space.* To appear in: Handbook of Teichmüller theory (ed. A. Papadopoulos).

[4] F. Herrlich and G. Schmithüsen: *A comb of origami curves in the moduli space M_3 with three dimensional closure.* To appear in Geom. dedicata.

[5] P. Hubert and S. Lelièvre: *Prime arithmetic Teichmüller discs in $H(2)$.* Israel Journal of Math. 151 (2006), 281–321.

[6] C. McMullen: *Billiards and Teichmüller curves on Hilbert surfaces.* J. Am. Math. Soc. 16, No.4 (2003), 857–885.

[7] M. Möller: *Variations of Hodge structure of Teichmüller curves.* Journal of the AMS 19 (2006), 327–344.

[8] G. Schmithüsen: *An algorithm for finding the Veech group of an origami.* Experim. Math. 13 (2004), 459–472.

[9] G. Schmithüsen: *Origamis with non congruence Veech groups.* This volume.

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