A foreshadow of principal bundles

Consider the the following map p from the unit circle S^1 to itself:

$$p : X = S^1 \to B = S^1, z \mapsto z^2 \text{ where } S^1 = \{z \in \mathbb{C} | |z| = 1\}$$

Let G be the group $(\mathbb{Z}/2\mathbb{Z}, +) = <\varphi>$ endowed with the discrete topology. Show that there is an action of G on S^1 from the right such that:

A) G acts effectively, i.e.: $\forall x \in S^1, s \in G : (x \cdot s = x \Rightarrow s = \bar{0})$.
Here $\bar{0}$ is the identity element in G.

B) X/G is isomorphic to B.

C) The map

$$\tau : X^* = \{(x, xs) | x \in X, s \in G\} \to G$$

with the property $x \cdot \tau(x, y) = y$ is continuous.

What is the fibre of a point on S^1?

Let now F be the unit interval $[-1, 1]$. Consider the action from the left of G by $\varphi : z \mapsto -z$ for the non trivial element $\varphi \in G$. This action and the action of on X from the right from above give an action on the product $X \times F$:

$$G \ni s : (x, t) \mapsto (x \cdot s, s^{-1} \cdot t).$$

Does the map $p_F : (X \times F)/G \to B, (x, t) \cdot G \mapsto p(x)$ also satisfy the properties A),B) and C) from above? The topological space $(X \times F)/G$ is a topological space which you know very well. Which one? What is the fibre of a point on B via p_F?

Getting explicit about group homology

Let G be a group. It holds

$$H_1(G; \mathbb{Z}) \cong \mathfrak{I}/\mathfrak{I}^2 \cong G/[G, G],$$

where $\mathfrak{I}$ is the so called *augmentation ideal*, i.e. the kernel of the ring homomorphism

$$\varphi \colon \mathbb{Z}[G] \to \mathbb{Z}, f \mapsto \sum_{x \in supp(f)} f(x).$$

Hint:
First show $H_1(G; \mathbb{Z}) \cong \mathfrak{I}/\mathfrak{I}^2$ by using the long exact sequence induced by the short exact sequence of $\mathbb{Z}[G]$-modules you get from φ. It is probably helpful to convince yourself that $\{g - 1 | g \in G\}$ is a basis of $\mathfrak{I}$ as a free $\mathbb{Z}$-module.
Again using this basis, you can show $\mathfrak{I}/\mathfrak{I}^2 \cong G/[G, G]$ elementary.

A useful property of the derived functor

Prove that the zeroth derived functors are isomorphic to the original functor, i.e. $R^0G(A) \cong G(A)$ and $L_0F(A) \cong F(A)$ for right exact functors F, left exact functors G and all modules A.