Exercises

Spectral Sequences

1 First Introduction

Definition: A *double complex* (or *bicomplex*) $C_{..}$ is a family $\{C_{pq}\}$ of modules with maps

$$d^h: C_{pq} \to C_{p-1,q} \text{ and } d^{\upsilon}: C_{pq} \to C_{p,q-1}$$

such that: $d^h \circ d^h = d^v \circ d^v = d^v d^h + d^h d^v = 0.$

Definition: The *total complexes* $Tot(C) = Tot^{\Pi}(C)$ and $Tot^{\oplus}(C)$ are defined as:

$$Tot^{\Pi}(C)_n = \prod_{p+q=n} C_{pq} \text{ and } Tot^{\oplus}(C)_n = \bigoplus_{p+q=n} C_{pq}$$

The formula $d = d^h + d^v$ defines maps

$$d: Tot^{\Pi}(C)_n \to Tot^{\Pi}(C)_{n-1} \text{ and } d: Tot^{\oplus}(C)_n \to Tot^{\oplus}(C)_{n-1}$$

with $d \circ d = 0$ and turns $Tot^{\Pi}(C)$ and $Tot^{\oplus}(C)$ into chain complexes.

The following part is nearly literally taken from Charles A. Weibel's »An introduction to homological algebra« and can be read at the beginning of chapter *5 Spectral Sequences*:

Now forget about the horizontal differentials and write $E_{pq}^0 := C_{pq}$. The columns of the double complex now form chain complexes with differentials d^v . We assume that the modules are zero unless p and q both nonnegative.

If we write E_{pq}^1 for the vertical homology $H_q(E_{p*}^0)$ at the (p,q) spot, we may once again arrange the data in a lattice, this time using the horizontal differentials d^h .

Now we write E_{pq}^2 for the horizontal homology $H_p(E_{*q}^1)$ at the (p, q) spot.

In a sense made clearer by the following exercises, the elements of E_{pq}^2 are a second-order approximation of the homology of $T_* = Tot(E_{**})$.

Exercise 1.1: Suppose that the double complex *E* consists solely of the two columns *p* and *p* – 1. Fix *n* and set q = n - p, so that an element of $H_n(T)$ is represented by an element $(a, b) \in E_{p-1,q+1} \times E_{pq}$. Show that we have calculated the homology of T = Tot(E) up to extension in the sense that there is a short exact sequence

$$0 \longrightarrow E^2_{p-1,q+1} \longrightarrow H_{p+q}(T) \longrightarrow E^2_{pq} \longrightarrow 0.$$

- **Exercise 1.2:** Show that E_{pq}^2 can be presented as the group of all pairs (a, b) in $E_{p-1,q+1} \times E_{pq}$ such that $0 = d^v b = d^v a + d^h b$, modulo the relation that these pairs are trivial: (a, 0); $(d^h x, d^v x)$ for $x \in E_{p,q+1}$; and $(0, d^h c)$ for all $c \in E_{p+1,q}$ with $d^v c = 0$.
 - If $d^{h}(a) = 0$, show that such a pair (a, b) determines an element of $H_{p+q}(T)$.
 - Show that the formula $d(a, b) = (0, d^h(a))$ determines a well-defined map $d: E_{pq}^2 \longrightarrow E_{p-2,q+1}^2$.

Exercise 1.3: Recall that we have assumed that E_{pq}^0 vanishes unless both $p \ge 0$ and $q \ge 0$. By diagram chasing, show that $E_{00}^2 = H_0(T)$ and that there is an exact sequence

$$H_2(T) \longrightarrow E_{20}^2 \longrightarrow E_{01}^2 \longrightarrow H_1(T) \longrightarrow E_{10}^2 \longrightarrow 0.$$

2 Exactness

Consider two modules D and E and morphisms such that the following diagram



is exact, i.e. kernel = image, at each vertex.

The map d = jk fulfills $d^2 = jkjk = j(kj)k = j0k = 0$ and therefore we may form the homology module H(E) = ker(d)/im(d). We get a second triangle



with morphisms $i' = i|_{i(D)}$, j' and k', which are given by j'(i(x)) = [j(x)] and k'([e]) = k(e) for $x \in D$ and $[e] \in H(E)$.

Exercise 2.1: Show that the maps i', j' and k' are well-defined and that this triangle, too, is exact at each vertex.